AMS 526 Sample Questions for Test 2

October 22, 2012

Note: The exam is closed-book. However, you can have a single-sided, one-page, letter-size cheat sheet.

1. Answer true or false and give a **brief justification**. (No credit without justification.)

   (a) The singular values of an orthogonal matrix are all ones.

   (b) When solving the least squares problem $\boldsymbol{Ax} \approx \boldsymbol{b}$ using a backward stable algorithm, if the perturbations in $\boldsymbol{A}$ is $O(\epsilon)$, then the error in the output is $O(\kappa(A)\epsilon)$.

   (c) If every singular value of a matrix $\boldsymbol{A}$ is zero, then $\boldsymbol{A} = \boldsymbol{0}$.

   (d) If $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ is symmetric positive definite, then $\boldsymbol{BAB}^T$ has the same eigenvalues as $\boldsymbol{A}$ for any nonsingular $\boldsymbol{B} \in \mathbb{R}^{m\times m}$.

2. Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with full rank, its pseudoinverse is $\boldsymbol{A}^+ = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$ if $m \geq n$ and is $\boldsymbol{A}^+ = \boldsymbol{A}^T(\boldsymbol{AA}^T)^{-1}$ if $m \leq n$. Express $\boldsymbol{A}^+$ based on its SVD for both cases.

3. Let $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix with Cholesky factorization $\boldsymbol{A} = \boldsymbol{R}^T\boldsymbol{R}$. Show that $\sqrt{\|\boldsymbol{A}\|_2} = \|\boldsymbol{R}\|_2$ and that $\sqrt{\kappa_2(\boldsymbol{A})} = \kappa_2(\boldsymbol{R})$, where κ_2 denotes the condition number measured in 2-norm.

4. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, where $m \geq n$, and $\boldsymbol{A}$ has full rank. Show that $\begin{bmatrix} \boldsymbol{I} & \boldsymbol{A} \\ \boldsymbol{A}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$ has a solution where $\boldsymbol{x}$ minimizes $\|\boldsymbol{Ax} - \boldsymbol{b}\|_2$. What is the condition number of the coefficient matrix in terms of the singular values of $\boldsymbol{A}$?

5. Given a linear least squares problem $\boldsymbol{Ax} \approx \boldsymbol{b}$ where $\boldsymbol{A}$ has more rows than columns and has full rank, name an efficient and numerically stable method for solving this problem. Explain why you favor this method over other alternatives.