AMS526: Numerical Analysis I  
(Numerical Linear Algebra)  
Lecture 10: Component-wise Sensitivity Analysis; Review

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Outline

1. Componentwise Sensitivity Analysis

2. Review for Midterm #1
The analysis we discussed so far are *normwise sensitivity analysis*, accompanied with *normwise backward error analysis*

Normwise sensitivity analysis is powerful, but estimated error may be overly pessimistic.

This over-estimation of error can be explained as follows:

For linear system $Ax = b$, we can rescale the problem as $$(WA\Omega)y = Wb, \quad \Omega y = x$$

If there exist diagonal matrices $W$ and $\Omega$, such that $WA\Omega$ is well conditioned, then $\delta x$ is small in terms of $\|\Omega^{-1}\delta x\|/\|\Omega^{-1}x\|$

However, the condition number of $A$ in $p$-norm, i.e., $\kappa_p(A) \neq \kappa_p(WA)$, so it can be inflated by poor scaling.

A better bound would be $\min \{\kappa_W(A) = \|A^{-1}\|_W \|A\|_W\}$, which we might call “minimum weighted-norm condition number”, but nobody knows how to compute it (yet).
Limitation of Normwise Sensitivity Analysis

- Another limitation of normwise backward error analysis is that small error in
  \[ \| \delta \mathbf{x} \| / \| \mathbf{x} \| \leq \epsilon \not\Rightarrow \| \delta x_{ij} \| / \| x_{ij} \| \leq \epsilon \]
  for each entry \( x_{ij} \)
- For example, suppose
  \[ A = \begin{bmatrix} 1.04 & 2.35 \\ 4.26 \times 10^{-6} & 6.32 \end{bmatrix} \text{ and } \delta A = \begin{bmatrix} 1.32 & 5.46 \\ 1.02 & 8.29 \end{bmatrix} \times 10^{-5}, \]
  then \( \| \delta A \|_{\infty} / \| A \|_{\infty} < 10^{-5} \) but \( |\delta a_{21}| / |a_{21}| > 1 \)
- In other words, normwise analysis can underestimate componentwise errors
- The dual aspects of overestimation and underestimation of normwise sensitivity analysis motivates the componentwise sensitivity analysis
Componentwise Sensitivity Analysis

- In *componentwise sensitivity analysis*, perturbations are considered small only if the perturbation in each component is small relative to that component, i.e.,
  \[
  \max_{i,j} \frac{|\delta a_{ij}|}{|a_{ij}|} \leq \epsilon
  \]

- To avoid issues with \(a_{i,j} \approx 0\), we say perturbation \(\delta A\) is *componentwise \(\epsilon\)-small* with respect to \(A\) if there is a positive \(\epsilon \ll 1\), such that
  \[
  |\delta a_{ij}| \leq \epsilon |a_{ij}| \quad \text{for} \quad i, j = 1, \ldots, n. \tag{1}
  \]

- Given matrix \(B = \{b_{ij}\}\), let \(|B|\) denote matrix \(|\{b_{ij}\}|\).
- We write \(|B| \leq |C|\) to mean that \(|b_{ij}| \leq |c_{ij}|\) for all \(i\) and \(j\)
- Then (1) can be rewritten as
  \[
  |\delta A| \leq \epsilon |A|.
  \]

- If \(A = BC\), then \(|A| \leq |B| |C|\).
Skeel Condition Number

- If \( A \in \mathbb{R}^{n \times n} \) is nonsingular, we define matrix \( K = |A^{-1}| |A| \)
- The **Skeel condition number** of \( A \) is

\[
\text{skell}(A) = \|K\|_\infty = \| |A^{-1}| |A| \|_\infty
\]

**Theorem**

Suppose \( A \in \mathbb{R}^{n \times n} \) is nonsingular, \( b \in \mathbb{R}^n \) is nonzero, and \( x \in \mathbb{R}^n \) is the unique solution of \( Ax = b \). Suppose \( \hat{x} = x + \delta x \) is the solution of \( A\hat{x} = b + \delta b \), where

\[
|\delta b| \leq \epsilon |b|.
\]

Then

\[
|\delta x| \leq \epsilon |A^{-1}| |A| |x|
\]

and

\[
\frac{||\delta x||_\infty}{||x||_\infty} \leq \epsilon \text{skell}(A)
\]
Sensitivity to Perturbation in $A$

**Theorem**

Suppose $A \in \mathbb{R}^{n \times n}$ is nonsingular, $b \in \mathbb{R}^n$ is nonzero, and $x \in \mathbb{R}^n$ is the unique solution of $Ax = b$. Suppose $\hat{x} = x + \delta x$ is the solution of $(A + \delta A)\hat{x} = b$, where

$$|\delta A| \leq \epsilon |A|.$$  

Then

$$|\delta x| \leq \epsilon |A^{-1}| |A| |\hat{x}|$$

and

$$\frac{\|\delta x\|_\infty}{\|\hat{x}\|_\infty} \leq \epsilon \text{skeel}(A).$$

If $\epsilon \text{skeel}(A) < 1$, then also

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq \frac{\epsilon \text{skeel}(A)}{1 - \epsilon \text{skeel}(A)}.$$
Iterative Refinement

- Componentwise backward stability is useful in analyzing iterative refinement.
- Let $\hat{x}$ denote an approximation to the solution of the system $Ax = b$.
- Let $\hat{r}$ be the associated residual: $\hat{r} = b - A\hat{x}$.
- If we solve the residual system $Az = \hat{r}$ exactly, then the vector $x = \hat{x} + z$ is the exact solution of $Ax = b$.
- In practice, $Az = \hat{r}$ is solved inexactly as $A\hat{z} = \hat{r}$, but $\hat{x} + \hat{z}$ may give a better solution than $\hat{x}$. 
Componentwise Backward Stability of Iterative Refinement

- Before 1980, it was believed that residual needs to be calculated in extended-precision for iterative refinement to be beneficial.
- In 1980, Skeel showed that if the system is not too badly conditioned and not too badly out of scale, then one step of iterative refinement is usually enough to ensure a componentwise backward stable solution.
- There are inexpensive methods for estimating $\text{skeel}(A)$, similar to those for estimating $\kappa_1(A)$.
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Fundamental Concepts

- Positive definiteness
- Sparsity of linear systems
- Norms and condition number of matrices
- Conditioning of problems
- Stability and backward stability of algorithms
- Efficiency of algorithms
Fundamental Algorithms

- Cholesky factorization $A = RR^T$
- $LDL^T$ factorization: $A = LDL^T$
- LU factorization (Gaussian elimination) with partial pivoting $AP = LU$
- Understand when they work, how they work, why they work, and how well they work
- Understand relationships among each other: how one transforms into another, and to make an intelligent choice