Outline

1 Orthogonal Vectors and Matrices

2 Projectors

3 Linear Least Squares Problems
Orthogonal Vectors

**Definition**

A pair of vectors are *orthogonal* if \( x^T y = 0 \).

In other words, the angle between them is 90 degrees.
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**Definition**
A set of nonzero vectors $S$ is *orthogonal* if they are pairwise orthogonal. They are *orthonormal* if it is orthogonal and in addition each vector has unit Euclidean length.
Orthogonal Vectors

**Theorem**

*The vectors in an orthogonal set $S$ are linearly independent.*

**Proof.**

Prove by contradiction. If a vector can be expressed as linear combination of the other vectors in the set, then it is orthogonal to itself.

**Question:** If the column vectors of an $m \times n$ matrix $A$ are orthogonal, what is the rank of $A$?

**Answer:**

$r = \min\{m, n\}$. In other words, $A$ has full rank.
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Components of Vector

- Given an orthonormal set \( \{q_1, q_2, \ldots, q_m\} \) forming a basis of \( \mathbb{R}^m \), vector \( v \) can be decomposed into orthogonal components as

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v = \sum_{i=1}^{m} (q_i^T v) q_i
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- \( q_i q_i^T \) is an orthogonal projection matrix. Note that it is NOT an orthogonal matrix.
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- More generally, given an orthonormal set \( \{q_1, q_2, \ldots, q_n\} \) with \( n \leq m \), we have
  \[
  v = r + \sum_{i=1}^{n} (q_i^T v) q_i = r + \sum_{i=1}^{n} (q_i q_i^T) v \text{ and } r^T q_i = 0, 1 \leq i \leq n
  \]

- Let \( Q \) be composed of column vectors \( \{q_1, q_2, \ldots, q_n\} \).
  \[
  QQ^T = \sum_{i=1}^{n} (q_i q_i^T)
  \] is an orthogonal projection matrix.
Orthogonal Matrices

**Definition**

A matrix is *orthogonal* if \( Q^T = Q^{-1} \), i.e., if \( Q^T Q = QQ^T = I \).

- In the real case, we say the matrix is *orthogonal*. Its column vectors are *orthonormal*.
- In other words, \( q_i^T q_j = \delta_{ij} \), the *Kronecker delta*.

Note: If \( Q \in \mathbb{C}^{m \times m} \), then \( Q \) is said to be *unitary* (instead of being orthogonal).
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**Question:** What is the geometric meaning of multiplication by an orthogonal matrix?

**Answer:** It preserves angles and Euclidean length. In the real case, multiplication by an orthogonal matrix $Q$ is a rotation (if $\det(Q) = 1$) or reflection (if $\det(Q) = -1$).
Outline

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2. Projectors
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A projector satisfies $P^2 = P$. They are also said to be idempotent.

- Orthogonal projector.
- Oblique projector.

Example: 

\[
\begin{bmatrix}
0 & 0 \\
\alpha & 1
\end{bmatrix}
\]

- is an oblique projector if $\alpha \neq 0$,
- is orthogonal projector if $\alpha = 0$. 
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Complementary Projectors

- Complementary projectors: $P$ vs. $I - P$.
- What space does $I - P$ project?

A projector separates $\mathbb{R}^m$ into two complementary subspaces: range space and null space (i.e., $\text{range}(P) + \text{null}(P) = \mathbb{R}^m$ and $\text{range}(P) \cap \text{null}(P) = \{0\}$ for projector $P \in \mathbb{R}^{m \times m}$). It projects onto range space along null space.

▶ In other words, $x = Px + r$, where $r \in \text{null}(P)$.
Complementary Projectors

- Complementary projectors: $P$ vs. $I - P$.
- What space does $I - P$ project?
  - Answer: $\text{null}(P)$.
  - $\text{range}(I - P) \supseteq \text{null}(P)$ because $Pv = 0 \Rightarrow (I - P)v = v$.
  - $\text{range}(I - P) \subseteq \text{null}(P)$ because for any $v$ $(I - P)v = v - Pv \in \text{null}(P)$.

- A projector separates $\mathbb{R}^m$ into two complementary subspace: range space and null space (i.e., $\text{range}(P) + \text{null}(P) = \mathbb{R}^m$ and $\text{range}(P) \cap \text{null}(P) = \{0\}$ for projector $P \in \mathbb{R}^{m \times m}$).

- It projects onto range space along null space
  - In other words, $x = Px + r$, where $r \in \text{null}(P)$

- Question: Are range space and null space of projector orthogonal to each other?
Orthogonal Projector

- An orthogonal projector is one that projects onto a subspace $S_1$ along a space $S_2$, where $S_1$ and $S_2$ are orthogonal.

Theorem

A projector $P$ is orthogonal if and only if $P = P^T$. 
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**Theorem**

A projector $P$ is orthogonal if and only if $P = P^T$.

**Proof.**

“If” direction: If $P = P^T$, then $(Px)^T(I - P)y = x^T(P - P^2)y = 0$.

“Only if” direction: Suppose $P$ projects onto $S_1$ along $S_2$ where $S_1 \perp S_2$, and $S_1$ has dimension $n$. Let $\{q_1, \ldots, q_n\}$ be orthonormal basis of $S_1$ and $\{q_{n+1}, \ldots, q_m\}$ be a basis for $S_2$. Let $Q$ be orthogonal matrix whose $j$th column is $q_j$, and we have $PQ = (q_1, q_2, \ldots, q_n, 0, \ldots, 0)$, so $Q^TPQ = \text{diag}(1, 1, \ldots, 1, 0, \ldots) = \Sigma$, and $P = Q\Sigma Q^T$. 
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Question: Are orthogonal projectors orthogonal matrices?
Basis of Projections

- **Projection with orthonormal basis**
  - Given any matrix \( \hat{Q} \in \mathbb{R}^{m \times n} \) whose columns are orthonormal, then \( P = \hat{Q} \hat{Q}^T \) is orthogonal projector, so is \( I - P \)
  - We write \( I - P \) as \( P_\perp \)
  - In particular, if \( \hat{Q} = q \), we write \( P_q = qq^T \) and \( P_{\perp q} = I - P_q \)
  - For arbitrary vector \( a \), we write \( P_a = \frac{aa^T}{a^Ta} \) and \( P_{\perp a} = I - P_a \)
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- **Projection with arbitrary basis**
  - Given any matrix $A \in \mathbb{R}^{m \times n}$ that has full rank and $m \geq n$
    \[
P = A(A^TA)^{-1}A^T\]
    is orthogonal projection
  - What does $P$ project onto?
Basis of Projections

- **Projection with orthonormal basis**
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- **Projection with arbitrary basis**
  - Given any matrix \( A \in \mathbb{R}^{m \times n} \) that has full rank and \( m \geq n \)
    \[
P = A(A^T A)^{-1} A^T
    \]
    is orthogonal projection
  - What does \( P \) project onto?
    - \( \text{range}(A) \)
    - \( (A^T A)^{-1} A^T \) is called the *pseudo-inverse* of \( A \), denoted as \( A^+ \)
Outline

1. Orthogonal Vectors and Matrices
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Linear Least Squares Problems

- Overdetermined system of equations $Ax \approx b$, where $A$ has more rows than columns and has full rank, in general has no solutions
- Example application: Polynomial least squares fitting
- In general, minimize the residual $r = b - Ax$
- In terms of 2-norm, we obtain linear least squares problem: Given $A \in \mathbb{R}^{m \times n}$, $m \geq n$, and $b \in \mathbb{R}^m$, find $x \in \mathbb{R}^n$ such that $\|b - Ax\|_2$ is minimized
- If $A$ has full rank, the minimizer $x$ is the solution to the normal equation
  \[ A^T Ax = A^T b \]
  or in terms of the pseudoinverse $A^+$,
  \[ x = A^+ b, \quad \text{where } A^+ = (A^T A)^{-1} A^T \in \mathbb{R}^{n \times m} \]
Geometric Interpretation

- $Ax$ is in range($A$), and the point in range($A$) closest to $b$ is its orthogonal projection onto range($A$)
- Residual $r$ is then orthogonal to range($A$), and hence $A^T r = A^T (b - Ax) = 0$
- $Ax$ is orthogonal projection of $b$, where $x = A^+ b$, so $P = AA^+ = A(A^T A)^{-1} A^T$ is orthogonal projection