Announcement: Final Exam

- Tuesday **12/11** between **8:30pm** and **11:00pm** in classroom Physics P122
- Final exam will be **accumulative** covering **all** the material from the semester
- About 50–60% will be on material after Test 2 (i.e., eigenvalue problems and iterative methods, which are closely connected with earlier materials, especially QR and SVD)
- As usual, you can have a **single-sided, one-page, letter-size** (8.5inx11in) cheat sheet
Topics Covered in The Course

- Fundamental concepts: norms, orthogonality, conditioning, stability
- Least squares problems using direct method (QR factorization)
- Singular value decomposition, properties, and relationship with eigenvalue problems
- Eigenvalue problems, properties, and algorithms (QR algorithm and Lanczos iterations)
- Solving linear systems using direct (Gaussian elimination) and iterative (Krylov subspace) methods
- Conditioning of problems, stability and backward stability of algorithms
- Efficiency of algorithms, convergence rate of iterative methods
Matrix Properties and Transformations

- **Properties**
  - Hermitian (symmetric), skew symmetry, positive definite
  - unitary (orthogonal), normal, (orthogonal and oblique) projection matrix
  - singular/nonsingular, defective/nondefective
  - triangular, Hessenberg, tridiagonal, diagonal, Jordan-form, sparse

- **Transformations**
  - orthogonalization (Gram-Schmidt)
  - triangularization (Gaussian elimination, Cholesky factorization, Householder QR)
  - reduction to Hessenberg or tridiagonal form
  - similarity transformation and unitary similarity transformation (Schur factorization)
  - congruence transformation (preserves symmetry and inertia)
Fundamental Algorithms

- QR factorization using classical and modified Gram-Schmidt
- QR factorization using Householder triangularization
- Gaussian elimination with partial pivoting and Cholesky factorization
- Reduction to Hessenberg/tridiagonal form for eigenvalue problems
- QR algorithm with or without shifts for eigenvalue problems
- Lanczos iterations and conjugate gradients
- Do not need to memorize the details of the algorithms
- Understand when they work, how they work, why they work, and how well they work
- Understand relationships among each other: how one transforms into another, and to make an intelligent choice
Eigenvalue Problem

- Eigenvalue problem of $m \times m$ matrix $A$ is $Ax = \lambda x$
- Characteristic polynomial is $\det(A - \lambda I)$
- Eigenvalue decomposition of $A$ is $A = \mathbf{X}\Lambda\mathbf{X}^{-1}$ (does not always exist)
- Geometric multiplicity of $\lambda$ is $\dim(\text{null}(A - \lambda I))$, and algebraic multiplicity of $\lambda$ is its multiplicity as a root of $p_A$, where algebraic multiplicity $\geq$ geometric multiplicity
- Similar matrices have the same eigenvalues, and algebraic and geometric multiplicities
- Schur factorization $A = QTQ^*$ uses unitary similarity transformations
Eigenvalue Algorithms

- Underlying concepts: power iterations, Rayleigh quotient, inverse iterations, convergence rate

- **Schur factorization** is typically done in two steps
  - Reduction to Hessenberg form for nonhermitian matrices or reduction to tridiagonal form for hermitian matrices by unitary similarity transformation
  - Finding eigenvalues of Hessenberg or **tridiagonal** form

- Finding eigenvalue of tridiagonal forms
  - QR algorithm with shifts, and their interpretations as (inverse) simultaneous iterations
  - Others: Bisection and divide-and-conquer

- Alternative method is Jacobi method for symmetric matrices using Jacobi rotations
Relationship between SVD and Eigenvalue Decomposition

- SVD works for all matrices (even rectangular matrices), but eigenvalue decomposition (i.e., diagonalization) works only for nondefective square matrices.
- Singular vectors are always orthonormal and singular values are always real, while eigenvectors may not be orthogonal and eigenvalues may be complex numbers.
- For normal matrices, singular values and eigenvalue are particularly closely related, which make them particularly powerful analytical tools.
Advantages and disadvantages of iterative methods vs. direct methods
We focus on Krylov subspace methods for symmetric matrices
Given \( A \) and \( b \), Krylov subspace is \( \{b, Ab, A^2b, \cdots A^kB\} \)

Key observation: QR factorization of leading vectors of Krylov subspace leads to Hessenberg form for nonsymmetric matrices and tridiagonal form for symmetric matrices

Lanczos iterations takes advantage of the tridiagonal form to get three-term recurrence version of Arnoldi iterations

**Conjugate gradient methods** for solving SPD linear systems: solution as quadratic optimization problem, finite-termination properties with exact arithmetic, and convergence with floating-point arithmetic

GMRES, Bi-CG, Bi-CGSTAB for nonsymmetric matrices

Concepts of preconditioners, and multigrid method