AMS 526 Homework 4, Fall 2014

Due: Wednesday 10/29 in class

1. (10 points)
   (a) Let $A \in \mathbb{R}^{n \times m}$ and $B = A^+ \in \mathbb{R}^{m \times n}$. Show that the following four relationships hold:
      i. $BAB = B$
      ii. $ABA = A$
      iii. $(BA)^T = BA$
      iv. $(AB)^T = AB$
   (b) Conversely, show that if $A$ and $B$ satisfy the above four conditions, then $B = A^+$.

2. (10 points) Let $A \in \mathbb{C}^{n \times n}$.
   (a) Use characteristic equations to show that $A$ and $A^T$ have the same eigenvalues.
   (b) Show that $\lambda$ is an eigenvalue of $A$ if and only if $\bar{\lambda}$ is an eigenvalue of $A^*$, where $A^*$ denotes the complex transpose of $A$.

3. (10 points) Suppose $A \in \mathbb{C}^{n \times n}$ is nonsingular, which implies that all eigenvalues of $A$ are nonzero. Show that if $v$ is an eigenvector of $A$ with associated eigenvalue $\lambda$, then $v$ is also an eigenvector of $A^{-1}$ with associated eigenvalue $\lambda^{-1}$.

4. (10 points) Suppose $A$ is similar to $B$. Show that if $A$ is nonsingular, then $B$ is also nonsingular, and $A^{-1}$ is similar to $B^{-1}$.

5. (15 points) A matrix $A \in \mathbb{C}^{n \times n}$ is skew Hermitian if $A^* = -A$.
   (a) (5 points) Prove that if $A$ is skew Hermitian and $B$ is unitarily similar to $A$, then $B$ is also skew Hermitian.
   (b) (5 points) Prove that the eigenvalues of a skew Hermitian matrix are purely imaginary.
   (c) (5 points) Do the above results apply to real matrices that are skew symmetric, i.e. $A \in \mathbb{R}^{n \times n}$ and $A^T = -A$? Why or why not?

6. (15 points) Suppose we have a $3 \times 3$ complex matrix and wish to introduce zeros by left- and/or right-multiplications by unitary matrices $Q_j$, such as Householder reflectors or Givens rotations. Consider the following three matrix structures:

   (i) $\begin{bmatrix} \times & \times & 0 \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$,  (ii) $\begin{bmatrix} \times & \times & 0 \\ 0 & 0 & \times \\ \times & \times & \times \end{bmatrix}$,  (iii) $\begin{bmatrix} \times & \times & 0 \\ 0 & 0 & \times \\ 0 & 0 & \times \end{bmatrix}$.

   For each one, decide which of the following situations holds, and justify your claim.
   (a) Can be obtained by a sequence of left-multiplications by matrices $Q_j$;
   (b) Not (a), but can be obtained by a sequence of left- and right-multiplications by matrices $Q_j$;
   (c) Cannot be obtained by a sequence of left- and right-multiplications by matrices $Q_j$.
7. (30 points) Starting from the template code, write a C code to call LAPACK routines dgesvd and dsyev to perform singular value decomposition and eigenvalue decomposition, and then use them to solve least squares problems $Ax \approx b$. Specifically, your code should use dgesvd to obtain the SVD of $A$ and apply Algorithm 1 given below to solve $Ax \approx b$. In addition, use dsyev to obtain the eigenvalues and eigenvectors of $A^TA$, and then use them to solve the normal equation $A^TAx = A^Tb$. You can find the interface definitions for dgesvd and dsyev by searching at http://netlib.org/cgi-bin/search.pl

Use your C code to solve the least squares problem arising from polynomial fitting of degree $n - 1$,

$$p_{n-1}(t) = x_1 + x_2t + x_3t^2 + \cdots + x_nt^{n-1},$$

from $m$ data points $(t_i, y_i)$, $m > n$. Let $t_i = (i - 1)/(m - 1)$, $i = 1, \ldots, m$, so that the data points are equally spaced on the interval $[0, 1]$. We will generate the corresponding values $y_i$ by first choosing values for the $x_j$, say $x_j = 1$, $j = 1, \ldots, n$, and evaluating the resulting polynomial to obtain $y_i = p_{n-1}(t_i)$, $i = 1, \ldots, m$. The objective is to see whether we can recover the $x_j$ that are used to generate $y_i$. Choose $n = 3, 4, \ldots, 15$ and $m = 2n$, and plot the 2-norm errors using SVD and eigenvalue decomposition. Submit your modified C code, the plots, and your conclusions of the comparative study.

Algorithm 1 Least Squares via SVD.

(a) Compute the reduced SVD $A = \hat{U}\hat{\Sigma}V^*$.
(b) Compute the vector $\hat{U}^*b$.
(c) Solve the diagonal system $\hat{\Sigma}w = \hat{U}^*b$ for $w$.
(d) Set $x = Vx$. 