AMS 526 Sample Final Exam

Total: 100 points

Note: The exam is closed-book. However, you can have a single-sided, one-page, letter-size cheat sheet.

1. (15 points) Answer true or false with a brief justification. (No credit without justification.)
   (a) If $A \in \mathbb{R}^{n \times n}$ is rank deficient, then its null space is orthogonal to its column space.
   (b) The eigenvalues of a real matrix are not necessarily real, but their product must be real.
   (c) In QR algorithm with shifts for finding eigenvalues of a symmetric matrix, using the Rayleigh quotient shift always leads to cubic convergence.
   (d) The Lanczos iteration applies only to symmetric and positive definite matrices.
   (e) The conjugate gradient method would converge slowly if all the eigenvalues of the matrix are equal.

2. (10 points) Order the following procedures from the least work required to the most work required, for a symmetric and indefinite matrix $A \in \mathbb{R}^{n \times n}$ with $n \gg 1$. Justify your answer.
   (a) LDL^T factorization with pivoting.
   (b) Gaussian elimination with partial pivoting.
   (c) Computing the inverse of A.
   (d) Computing all the eigenvalues of A, suppose A is symmetric and tridiagonal.

3. (10 points) Given a matrix $A \in \mathbb{C}^{m \times n}$ with full rank, its pseudoinverse is $A^+ = (A^*A)^{-1}A^*$ if $m \geq n$ and is $A^+ = A^*(AA^*)^{-1}$ if $m \leq n$. Express A^+ based on its SVD for both cases.

4. (15 points) Assume $A \in \mathbb{R}^{m \times n}$ has full rank, where $m \geq n$. Suppose A is composed of a subset of rows of B, i.e., $B = \begin{bmatrix} A \\ C \end{bmatrix}$, where $C \in \mathbb{R}^{k \times n}$ for some $k \geq 1$.
   (a) (10 points) Show that $\|A\|_p \leq \|B\|_p$ for any $p \in [1, \infty]$.
   (b) (5 points) Suppose $A = Q_1R_1$ and $B = Q_2R_2$ be the reduced QR factorizations of A and B, respectively. Show that $\|R_1\|_2 \leq \|R_2\|_2$.

5. (10 points) Let $A \in \mathbb{R}^{m \times n}$, where $m \geq n$, and A has full rank. Show that $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ has a solution, where x minimizes $\|Ax - b\|_2$.

6. (10 points) Suppose $A \in \mathbb{R}^{n \times n}$ is normal, and x is an eigenvector corresponds to a complex eigenvalue λ_i with a nonzero imaginary parts. Show that $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ are orthogonal to each other, and $\|\mathrm{Re}(x)\| = \|\mathrm{Im}(x)\|$. (Hint: The eigenvalues and eigenvectors of a real matrix appear in conjugate pairs.)

7. (10 points) Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, and $B \in \mathbb{R}^{n \times n}$ is symmetric. Show that the matrix BA is diagonalizable, and all its eigenvalues are real.

8. (10 points) Suppose $A \in \mathbb{C}^{m\times m}$ is Hermitian. Given a unit length vector $q \in \mathbb{C}^m$, prove or disprove the following statement: There exists a unitary matrix $Q \in \mathbb{C}^{m\times m}$ whose **last** column is q, such that Q^*AQ is a tridiagonal matrix.

9. (10 points) Suppose $A \in \mathbb{R}^{m\times m}$ is symmetric and positive definite. In the conjugate gradient method for solving $Ax = b$, show that the subspace spanned by the first n search directions is the same as the Krylov subspace $\mathcal{K}_n = \langle b, Ab, A^2b, \ldots, A^{n-1}b\rangle$.