AMS 526 Homework 1

Due: Wednesday 09/12 in class.

1. (10 points) Consider matrices A, X, and B, partitioned as indicated.

$$A = \left[\begin{array}{c|cc} 1 & 3 & 2 \\ 2 & 1 & 1 \\ \hline -1 & 0 & 1 \end{array}\right] \quad X = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 2 & 1 & 1 \\ -1 & 2 & 0 \end{array}\right] \quad B = \left[\begin{array}{c|cc} 5 & 7 & 4 \\ 3 & 3 & 3 \\ \hline -2 & 2 & -1 \end{array}\right]$$

   Thus, for example, $A_{12} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $A_{21} = \begin{bmatrix} -1 \end{bmatrix}$. Compute $B_{ij} = A_{i1}X_{1j} + A_{i2}X_{2j}$ for $i = 1, 2$ and $j = 1, 2$ to verify that block-matrix multiplication gives the same result as matrix multiplication for $B = AX$.

2. (15 points) Show that if R is a nonsingular $n \times n$ upper-triangular matrix, then R^{-1} is also upper-triangular. Analogously, if L is a nonsingular $n \times n$ lower-triangular matrix, then L^{-1} is also lower-triangular.

3. (10 points) Verify that $\|xy^H\|_F = \|xy^H\|_2 = \|x\|_2\|y\|_2$ for any $x, y \in \mathbb{C}^n$.

4. (15 points) If u and v are in $\mathbb{C}^n$, then the matrix $A = I + uv^H$ is called a *rank-one perturbation to the identity matrix.* Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^H$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is null(A)?

5. (20 points) For each of the following verify the inequality and give an example of a nonzero vector or matrix (for general m and n) for which equality is achieved. In this problem x is an n-vector and A is $m \times n$ matrix.

   (a) $\|x\|_\infty \leq \|x\|_2$

   (b) $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$

   (c) $\|A\|_\infty \leq \sqrt{n}\|A\|_2$

   (d) $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$

6. (30 points) Implement two MATLAB functions to perform ABx in two ways: $(AB)x$ and $A(Bx)$, where $A, B \in \mathbb{R}^{n\times n}$. **You must write your own functions of matrix-matrix or matrix-vector multiplications using for loops and scalar operations; do NOT use MATLAB's built-in function for matrix-matrix or matrix-vector multiplications**. Use MATLAB's built-functions `tic` and `toc` to measure the runtimes for each n, and plot the runtimes of two functions against n for $n = 50 \times [1, 2, 4, 8, 16]$. Submit your completed code and plots to the TA by email. Also submit a report of an analysis of the numbers of floating point operations required by the two approaches. Does your analysis correlate well with their actual performances?
   (Note: If you are unfamiliar with MATLAB, you can start with the template file on class webpage.)