AMS 526 Homework 3, Fall 2018

Due: Monday 10/22 in class

1. (15 points) Show that for Gaussian elimination with partial pivoting applied to any matrix $A \in \mathbb{R}^{n\times n}$, the growth factor $\rho = \max_{i,j} |u_{ij}| / \max_{i,j} |a_{ij}|$ satisfies $\rho \leq 2^{n-1}$.

2. (15 points) Carefully prove by induction that the Cholesky decomposition is unique: Suppose $A = R^T R = S^T S$, where R and S are both upper-triangular matrices with positive main-diagonal entries. Partition A, R and S conformably and prove that the parts of S must equal the corresponding parts of R.

3. (15 points) Determine the (a) eigenvalues, (b) determinant, and (c) singular values of a Householder reflector.

4. (15 points)

   (a) (10 points) Let orthogonal matrices $Q_1 \ldots Q_k \in \mathbb{R}^{m\times m}$ be fixed and consider the problem of computing the product $B = Q_k \cdots Q_1 A$, where $A \in \mathbb{R}^{m\times n}$. Let the computation be carried out from right to left by straightforward floating point operations on computer satisfying both axioms. Show that this algorithm is backward stable. (Here A is thought of as data that can be perturbed; the matrices Q_j are fixed and not to be perturbed.)

   (b) (5 points) Give an example to show that this result no longer holds if the orthogonal matrices Q_j are replaced by arbitrary matrices $X_j \in \mathbb{R}^{m\times m}$.

5. (10 points) Consider the example

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1.0001 \\ 2 & 1.0001 \end{bmatrix}, \qquad b = \begin{bmatrix} 4 \\ 0.0001 \\ 4.0001 \end{bmatrix}.$$

   (a) What are the matrices A^+ and $P = A^+ A$ for this example? Give exact answers.

   (b) Find the exact solutions x and $y = Ax$ to the least squares problem $Ax \approx b$.

6. (30 points) In this exercise, you need to implement the Householder QR factorization, the implicit calculation of product $\hat{Q}^T b$, and back substitution. Then, use these algorithms as building blocks to solve the following least squares problem arising from polynomial fitting. We fit a polynomial of degree $n-1$,

$$p_{n-1}(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1},$$

   to m data points (t_i, y_i), $m > n$. Let $t_i = (i-1)/(m-1)$, $i = 1, \ldots m$, so that the data points are equally spaced on the interval $[0, 1]$. We will generate the corresponding values y_i by first choosing values for the x_j, say $x_j = 1$, $j = 1, \ldots, n$, and evaluating the resulting polynomial to obtain $y_i = p_{n-1}(t_i)$, $i = 1, \ldots, m$. Our objective is to see whether we can recover the x_j that are used to generate y_i, and measure the error as the difference between the computed x_j and the exact x_j.

   In addition, use the modified Gram-Schmidt procedure to solve the same least squares problem. Compare the two methods in terms of accuracy and efficiency.

Implement the algorithms using MATLAB. You must implement Householder QR and modified Gram-Schmidt procedures yourself, instead of using MATLAB's built-in functions. Choose $n = 3, 4, \ldots, 15$ and $m = 2n$, and plot the 2-norm errors using Householder and modified Gram-Schmidt algorithms. Submit your MATLAB code, the plots, and your conclusions of the comparative study.