AMS 526 Homework 4, Fall 2018

Due: Wednesday 11/05 in class

1. (10 points)

   (a) Let $A \in \mathbb{R}^{n\times m}$ and $B = A^+ \in \mathbb{R}^{m\times n}$. Show that the following four relationships hold:

      i. $BAB = B$

      ii. $ABA = A$

      iii. $(BA)^T = BA$

      iv. $(AB)^T = AB$

   (b) Conversely, show that if A and B satisfy the above four conditions, then $B = A^+$.

2. (10 points) Suppose A is similar to B. Show that if A is nonsingular, then B is also nonsingular, and A^{-1} is similar to B^{-1}.

3. (10 points) Suppose that $A \in \mathbb{C}^{n\times n}$ is normal, i.e., $AA^* = A^*A$.

   (a) Show that if A is also triangular, then it must be diagonal.

   (b) Show that a matrix is normal if and only if it has n orthonormal eigenvectors.

4. (15 points) A matrix $A \in \mathbb{C}^{n\times n}$ is *skew Hermitian* if $A^* = -A$.

   (a) (5 points) Prove that if A is skew Hermitian and B is unitarily similar to A, then B is also skew Hermitian.

   (b) (5 points) Prove that the eigenvalues of a skew Hermitian matrix are purely imaginary.

   (c) (5 points) Do the above results apply to real matrices that are *skew symmetric*, i.e. $A \in \mathbb{R}^{n\times n}$ and $A^T = -A$? Why or why not?

5. (15 points) Suppose we have a 3×3 complex matrix and wish to introduce zeros by left- and/or right-multiplications by unitary matrices Q_j, such as Householder reflectors or Givens rotations. Consider the following three matrix structures:

$$(i)\begin{bmatrix} \times & \times & 0 \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}, \quad (ii)\begin{bmatrix} \times & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{bmatrix}, \quad (iii)\begin{bmatrix} \times & \times & 0 \\ 0 & 0 & \times \\ 0 & 0 & \times \end{bmatrix}.$$

   For each one, decide which of the following situations holds, and justify your claim.

   (a) Can be obtained by a sequence of left-multiplications by matrices Q_j;

   (b) Not (a), but can be obtained by a sequence of left- and right-multiplications by matrices Q_j;

   (c) Cannot be obtained by a sequence of left- and right-multiplications by matrices Q_j.

6. (10 points) Let $A \in \mathbb{C}^{m\times m}$ be given, not necessarily Hermitian. Show that a number $z \in \mathbb{C}$ is a Rayleigh quotient of A if and only if it is a diagonal entry of Q^*AQ for some unitary matrix Q. Thus Rayleigh quotients are just diagonal entries of matrices, once you transform orthogonally to the right coordinate system.

7. (30 points) MATLAB Programming. Implement the following simple version of QR iteration with shifts for computing the eigenvalues of a general real matrix A.
Repeat until convergence:
1. $\mu = a_{n,n}$ (use corner entry as shift)
2. Compute QR factorization $QR = A - \mu I$
3. $A = RQ + \mu I$

You can implement using the built-in `qr` function in MATLAB for your implementation. Note that you need to deflate the matrix to a smaller system if the off-diagonal entries are close to zero. After computing the eigenvalues, also compute the eigenvectors.

Test your code for the matrices $A = \begin{bmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Also, solve the problem using the MATLAB built-in function `eig` to compute all of the eigenvalues and eigenvectors of the matrix, and compare the results with those obtained from your implementation. Submit the code and a brief report describing your algorithm and the results.

Hint: For these 3×3 matrices, you can implement deflation as follows. At the kth step, suppose

$$A^{(k)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

If $|a_{13}|, |a_{23}|, |a_{31}|$ and $|a_{32}|$ are all close to 0 (smaller than some ϵ close to machine epsilon), then set them to zero and apply QR iteration with shift to

$$A^{(k)}_{1:2,1:2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

to find the remaining two eigenvalues. After finding the eigenvalues, you can estimate their corresponding eigenvectors using Rayleigh-quotient iteration.