AMS 526 Homework 5, Fall 2018

Due on Monday 11/19th in class

1. (10 points) The preliminary reduction to tridiagonal form would be of little use if the steps of the QR algorithm did not preserve this structure. Fortunately, they do.

   (a) In the QR factorization $A = QR$ of a symmetric tridiagonal matrix A, which entries of R are in general nonzero? Which entries of Q? (In practice we do not form Q explicitly.)

   (b) Show that the tridiagonal structure is recovered when the product is RQ is formed.

2. (10 points) Suppose the largest off-diagonal entry is annihilated at each step of the Jacobi algorithm for a matrix $A \in \mathbb{R}^{n\times n}$. Show that the sum of the squares of the off-diagonal entries decreases by at least the factor $1 - 2/(n^2 - n)$ at each step?

3. (10 points) Let $A \in \mathbb{C}^{n\times n}$ be Hermitian ($A = A^*$), and let x be a right eigenvector associated with eigenvalue λ. Assuming λ is a distinct eigenvalue, show that the condition number for computing the eigenvalue λ is $\kappa = 1$.

4. (20 points)

   (a) (10 points) Let $A \in \mathbb{C}^{n\times n}$ be tridiagonal and Hermitian, with all its sub- and super-diagonal entries nonzero. Show that the eigenvalues of A are distinct. (Hint: Show that for any $\lambda \in \mathbb{C}$, $A - \lambda I$ has rank at least $n - 1$.)

   (b) (10 points) Show that if the entries of both principal diagonals of a bidiagonal matrix are all nonzero, then the singular values of the matrix are distinct.

5. (20 points) Given $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n\backslash\{0\}$, consider the Arnoldi iteration

$$AQ_k = Q_{k+1}\tilde{H}_k,$$

   where $\tilde{H}_k \in \mathbb{R}^{(k+1)\times k}$ is upper Hessenberg, and Q_k is composed of orthonormal columns with $q_1 = b/\|b\|$.

   (a) (10 points) Show that if A is symmetric, then $\tilde{H}_k$ is tridiagonal.

   (b) (10 points) Show that $\text{span}\{q_1, q_2, \dots q_k\}$ is equal to the Krylov subspace $\mathcal{K}_k(A, b) = \text{span}\{b, Ab, \dots, A^{k-1}b\}$, given that the dimension of $\mathcal{K}_k(A, b)$ is equal to k.

6. (30 points) Starting from the template code, write a C code to call LAPACK routines `dgesvd` and `dsyev` to perform singular value decomposition and eigenvalue decomposition, and then use them to solve least squares problems $Ax \approx b$. Specifically, your code should use `dgesvd` to obtain the SVD of A and apply Algorithm 1 given below to solve $Ax \approx b$. In addition, use `dsyev` to obtain the eigenvalues and eigenvectors of A^TA, and then use them to solve the normal equation $A^TAx = A^Tb$. You can find the interface definitions for `dgesvd` and `dsyev` by searching at `http://netlib.org/cgi-bin/search.pl`. Use your C code to solve the least squares problem arising from polynomial fitting of degree $n - 1$,

$$p_{n-1}(t) = x_1 + x_2t + x_3t^2 + \cdots + x_nt^{n-1},$$

from m data points (t_i, y_i), $m > n$. Let $t_i = (i-1)/(m-1)$, $i = 1, \ldots m$, so that the data points are equally spaced on the interval $[0, 1]$. We will generate the corresponding values y_i by first choosing values for the x_j, say $x_j = 1$, $j = 1, \ldots, n$, and evaluating the resulting polynomial to obtain $y_i = p_{n-1}(t_i)$, $i = 1, \ldots, m$. The objective is to see whether we can recover the x_j that are used to generate y_i. Choose $n = 3, 4, \ldots, 15$ and $m = 2n$, and plot the 2-norm errors using SVD and eigenvalue decomposition. Submit your modified C code, the plots, and your conclusions of the comparative study.

Algorithm 1 Least Squares via SVD.

(a) Compute the reduced SVD $A = \hat{U}\hat{\Sigma}V^*$.

(b) Compute the vector $\hat{U}^*b$.

(c) Solve the diagonal system $\hat{\Sigma}w = \hat{U}^*b$ for w.

(d) Set $x = Vx$.
