AMS 527, Spring 2019, Homework 4

100 points. Due: Monday 03/25

Electronic submission is required for the programming problems. Please email your programs and the report to the TA. Your email should have the subject line “AMS527 HW#4 Submission”. For the written part, you are encouraged (but not required) to typeset using \LaTeX or LyX (an easy-to-use front-end of \LaTeX). For electronic submission, homework is due at 11:59pm on the due date. For paper submission, homework is due in class on the due date.

1. (10 points) Exercise 8.5 on page 376 of Heath book.
   (a) If the integrand $f$ is twice continuously differentiable and $f''(x) \geq 0$ on $[a,b]$, show that the composite midpoint and trapezoid quadrature rules satisfy the bracketing property
   
   $$M_k(f) \leq \int_a^b f(x) \, dx \leq T_k(f).$$

   (b) If the integrand $f$ is convex on $[a,b]$ (see Section 6.2.1), show that the composite midpoint and trapezoid quadrature rules satisfy the bracketing property in part a.

2. (10 points) Exercise 8.6 on page 376 of Heath book.
   Suppose that Lagrange interpolation at a given set of nodes $x_1, x_2, \ldots, x_n$ is used to derive a quadrature rule. Prove that the corresponding weights are given by the integrals of the Lagrange basis functions,
   
   $$w_i = \int_a^b l_i(x) \, dx, \quad i = 1, \ldots, n.$$

3. (10 points) Exercise 8.9 on page 376 of Heath book.
   Let $p$ be a real polynomial of degree $n$ such that
   
   $$\int_a^b p(x)x^k dx = 0, \quad k = 0, \ldots, n - 1.$$

   (a) Show that the $n$ zeros of $p$ are real, simple and lie in the open interval $(a,b)$. (Hint: Consider the polynomial $q_k(x) = (x-x_1)(x-x_2)\cdots(x-x_k)$, where $x_i, \quad i = 1, \ldots, k$, are the roots of $p$ in $[a,b]$.)
   (b) Show that the $n$-point interpolatory quadrature rule on $[a,b]$ whose nodes are the zeros of $p$ has degree $2n - 1$. (Hint: Consider the quotient and remainder polynomials when a given polynomial is divided by $p$.)

   The forward difference formula
   
   $$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

   and the backward difference formula
   
   $$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

   are both first-order accurate approximations to the first derivative of a function $f : \mathbb{R} \to \mathbb{R}$. What order accuracy results if we average these two approximations? Support your answer with an error analysis.
5. (10 points) Exercise 8.15 on page 377 of Heath book. Archimedes approximated the value of \(\pi\) by computing the perimeter of a regular polygon inscribing or circumscribing a circle of diameter 1. The perimeter of an inscribed polygon with \(n\) sides is given by
\[
p_n = n \sin(\pi/n),
\]
and that of circumscribed polygon by
\[
q_n = n \tan(\pi/n),
\]
and these values provide lower and upper bounds, respectively, on the value of \(\pi\).

(a) Using the power series expansions for the sine and tangent functions, show that \(p_n\) and \(q_n\) can be expressed in the form
\[
p_n = a_0 + a_1 h^2 + a_2 h^4 + \cdots
\]
and
\[
q_n = b_0 + b_1 h^2 + b_2 h^4 + \cdots,
\]
where \(h = 1/n\). What are the true values of \(a_0\) and \(b_0\)?
(b) Given the values of \(p_6 = 3.0000\) and \(p_{12} = 3.1058\), use Richardson extrapolation to produce a better estimate for \(\pi\). Similarly, given the values \(q_6 = 3.4641\) and \(q_{12} = 3.2154\), use Richardson extrapolation to produce a better estimate for \(\pi\).

\[
\int_{0}^{1} \frac{4}{1 + x^2} \, dx = \pi,
\]
one can compute an approximate value for \(\pi\) using numerical integration of the given function.

(a) Use the midpoint, trapezoid and Simpson composite quadrature rules to compute the approximate value for \(\pi\) in this manner for various step size \(h\). Try to characterize the error as a function of \(h\) for each rule, and also compare the accuracy of the rules with each other (based on the known value of \(\pi\)). Is there any point beyond which decreasing \(h\) yields no further improvement? Why?
(b) Implement Romberg integration and repeat part a using it.
(c) Compute \(\pi\) again by the same method, this time using a library routine for adaptive quadrature (such as \texttt{quad} in MATLAB) and various error tolerances. How reliable is the error estimate it produces? Compare the work required (integrand evaluations and elapsed time) with that for parts a and b.
(d) Compute \(\pi\) again by the same method, this time using Monte Carlo integration with various numbers \(n\) of sample points. Try to characterize the error as a function of \(n\), and also compare the work required with that for the previous methods. For a suitable random number generator, see Section 13.4.

Using any method you choose, evaluate the double integral
\[
\int \int e^{-xy} \, dx \, dy
\]
over each of the following regions:
(a) The unit square, i.e., \(0 \leq x \leq 1, 0 \leq y \leq 1\).
(b) The quarter of the unit disk lying in the first quadrant, i.e., \(x^2 + y^2 \leq 1, x \geq 0, y \geq 0\).