AMS 527, Spring 2019, Homework 7

100 points. Due: Wednesday 05/08

Electronic submission is required for the programming problems. Please email your programs and the report to the TA. Your email should have the subject line “AMS527 HW#7 Submission”. For the written part, you are encouraged (but not required) to typeset using \LaTeX or LyX (an easy-to-use front-end of \LaTeX). For electronic submission, homework is due at 11:59pm on the due date. For paper submission, homework is due in class on the due date.


Suppose you are given a general-purpose subroutine for solving initial value problems for systems of $n$ first order ODEs $y' = f(t,y)$, and this is the only software tool you have available. For each type of problem in parts a, b, and c to follow, describe how you could use this routine to solve it. In each case, your answer should address the following points:

1. What is the function $f$ for the ODE subproblem?
2. How would you obtain the necessary initial conditions?
3. What special properties, if any, would the ODE subproblem have that would affect the choice of ODE method?

(a) Compute the definite integral
$$\int_{a}^{b} g(s) \, ds$$

(b) Solve the two-point boundary value problem
$$y'' = y^2 + t, \quad 0 \leq t \leq 1,$$
with boundary conditions
$$y(0) = 0, y(1) = 1.$$  

(c) Solve the heat equation
$$u_t = cu_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0,$$
with initial condition
$$u(0, x) = g(x), \quad 0 \leq x \leq 1,$$
and boundary conditions
$$u(t, 0) = 0, \quad u(t, 1) = 0, \quad t \geq 0.$$  

2. (15 points) Exercise 11.3 on page 490 of Heath book.

Give an example to show that neither consistency nor stability alone is sufficient to guarantee convergence of a finite difference scheme for solving a partial differential equation numerically.

3. (10 points) Given the Poisson equation in 2-D
$$u_{xx} + u_{yy} = f(x, y), \quad 0 \leq x, y \leq 1,$$
with homogeneous boundary conditions $u(x, y) = 0$. Approximate its solution by $u(x, y) \approx \sum_{i=1}^{n} s_i \phi_i(x, y)$, where $\phi_i(t)$ is the basis function.
(a) Use Galerkin’s method to derive the weak form

\[- \sum_{j=1}^{n} \int_{S} \frac{\partial \phi_{i}(x,y)}{\partial x} \frac{\partial \phi_{j}(x,y)}{\partial x} + \frac{\partial \phi_{i}(x,y)}{\partial y} \frac{\partial \phi_{j}(x,y)}{\partial y} \, dx \, dy = \int_{0}^{1} \int_{0}^{1} f(x,y) \phi_{i}(x,y) \, dx \, dy \]  

(1)

What conditions does \( \phi_{j}(x,y) \) need to satisfy for the weak form to be valid?

(b) Show how to use Rayleigh-Ritz method (i.e., minimum principle) to derive the weak form (1).


(a) Show that the matrix \( (1/\sqrt{n})F_{n} \) is unitary, where \( F_{n} \) is the Fourier matrix of order \( n \).

(b) Using this result, prove the discrete form of Parseval’s Theorem,

\[ \|y\|_{2}^{2} = n \|x\|_{2}^{2}, \]

where \( y = \text{DFT}(x) \).

5. (15 points) FFT Exercise 12.10 on page 508 of Heath book.

The standard second-order centered finite difference approximation to the Poisson equation in one dimension with periodic Dirichlet boundary conditions yields an \( n \times n \) matrix of the form

\[ A = \begin{bmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & \cdots & 0 & -1 & 2 \end{bmatrix}. \]

Show that each column of the Fourier matrix \( F_{n} \) is an eigenvector of \( A \), which implies that \( F_{n}^{-1}AF_{n} \) is diagonal. This fact forms the basis for fast direct methods for the Poisson equation called “fast Poisson solvers”.


(a) Use the method of lines and an ODE solver of your choice to solve the heat equation

\[ u_{t} = u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0, \]

with initial condition

\[ u(0,x) = \sin(\pi x), \quad 0 \leq x \leq 1, \]

and Dirichlet boundary conditions

\[ u(t,0) = 0, \quad u(t,1) = 0, \quad t \geq 0. \]

Integrate from \( t = 0 \) to \( t = 0.1 \). Plot the computed solution, preferably as a three dimensional surface over the \( (t,x) \) plane. If you do not have three-dimensional plotting capacity, plot the solution as a function of \( x \) for a few values of \( t \), including the initial and final times. Determine the maximum error in the computed solution by comparing with the exact solution

\[ u(t,x) = \exp(-\pi^{2}t) \sin(\pi x). \]

Experiment with various spatial mesh sizes \( \Delta x \), and try to characterize the error as a function of \( \Delta x \). On a log-log scale, plot the maximum error as a function of \( \Delta x \).

(b) Repeat part a, but this time with initial condition

\[ u(0,x) = \cos(\pi x), \quad 0 \leq x \leq 1, \]

and Neumann boundary conditions

\[ u_{x}(t,0) = 0, \quad u_{x}(t,1) = 0, \quad t \geq 0, \]

and compare with the exact solution

\[ u(t,x) = \exp(-\pi^{2}t) \cos(\pi x). \]