AMS 527 Sample Test 2

1. Answer whether the following statements are true or false and give a brief explanation.
   
   (a) When using the same number of nodes, Gaussian quadrature rule is in general more accurate than Newton-Cotes quadrature rule.
   
   (b) Because evaluating a definite integral is well-conditioned problem, the quadrature rules are always stable.
   
   (c) When using the same number of subintervals, the composite midpoint rule is generally more accurate than the composite trapezoid rule.
   
   (d) In solving an initial value problem for an ODE numerically, the global error is always proportional to the sum of the local errors.
   
   (e) For approximating a stable solution of an ODE numerically, an implicit method is always stable.

2. If \( Q(f) = \sum_{i=1}^{n} w_i f(x_i) \) is an interpolatory quadrature rule (i.e., based on polynomial interpolation) on the interval \([-1, 1]\), then what is the value of \( \sum_{i=1}^{n} w_i \)? Justify your answer.

3. The centered difference formula for the second derivative is given by
\[
f''(x) \approx \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} \tag{1}
\]

   (a) (6 points) Use Taylor series expansion to derive the forward difference formula
   
   \[
f'(x) = \frac{f(x + h) - f(x)}{h} + \frac{f''(x)}{c_1} h + \frac{f'''(x)}{c_2} h^2 + \frac{f^{(4)}(x)}{c_3} h^3 + O(h^4) \approx \frac{f(x + h) - f(x)}{h} \]

   and the backward difference formula
   
   \[
f'(x) = \frac{f(x) - f(x - h)}{h} + \frac{f''(x)}{c_4} h + \frac{f'''(x)}{c_5} h^2 + \frac{f^{(4)}(x)}{c_6} h^3 + O(h^4) \approx \frac{f(x) - f(x - h)}{h} \]

   What are the coefficients \( c_i, i = 1, 2, \ldots, 6 \)?

   (b) (9 points) Combine the above two formulas to derive the center difference formula (1). What is the order of accuracy of this approximation to the second derivative?

4. Write the following ODE as an equivalent first-order system of ODEs:
\[y''' = -yy''.\]

5. Consider the initial value problem for the ODE
\[y' = -y^2 \]

   with the initial condition \( y(0) = 1 \).

   (a) Take one-step of Euler’s method with step size 0.1 to compute the solution \( y_1 \) at time \( t_1 = 0.1 \).
   
   (b) Write down the algebraic equation for one-step of backward Euler method with step size 0.1 to compute the solution \( y_1 \) at time \( t_1 = 0.1 \).
   
   (c) Is the step size 0.1 within the stability range for Euler’s method and the backward Euler method?