AMS 527 Sample Final Exam

May 10, 2009

- Basic concept questions: Go through the review questions in textbook.
- Review questions in homework, test1, test 2, and sample questions listed in the review notes.

The following questions are in addition to the above questions.

1. Consider the boundary value problem

$$-u'' + (2-t)u = t,\ 0 < t < 1$$

with boundary conditions

$$u(0) = u(1) = 0.$$

Solve it using a finite difference method with centered difference approximation to u'' with a mesh size $h = \frac{1}{4}$ (i.e., $n = 4$). Note that it suffices for you to arrive at the linear system; you do not need to solve the linear system.

2. Consider the Poisson's equation in 2-D

$$u_{xx} + u_{yy} = f(x, y), \quad 0 < x, y < 1,$$

with boundary conditions

$$\begin{aligned} u(x, y) &= 0, \quad x = 0, 1. \\ u(x, y) &= 1, \quad y = 0, 1. \end{aligned}$$

Approximate its solution by $u(x, y) \approx \sum_{i=1}^{n} s_i \phi_i(x, y)$, where $\phi_i(t)$ is the basis function.

(a) Show how to use Galerkin's method to derive the weak form

$$-\sum_{j=1}^{n} \iint \frac{\partial \phi_i(x,y)}{\partial x}\frac{\partial \phi_j(x,y)}{\partial x} + \frac{\partial \phi_i(x,y)}{\partial y}\frac{\partial \phi_j(x,y)}{\partial y}\, dx\, dy = \iint_0^1 f(x,y)\,\phi_i(x,y)\, dx\, dy \qquad (1)$$

What conditions does $\phi_j(x, y)$ need to satisfy for the weak form to be valid?

(b) Show how to use Rayleigh-Ritz method (i.e., minimum principle) to derive the weak form (1).

3. All entries in the factorization of Fourier matrix $\boldsymbol{F}_6$ involves powers of $\omega = e^{-2\pi i/6}$, the sixth root of 1:

$$\boldsymbol{F}_6 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{D} \\ \boldsymbol{I} & -\boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_3 & \\ & \boldsymbol{F}_3 \end{bmatrix} \boldsymbol{P}_6^T,$$

where $\boldsymbol{P}_6$ is an even-odd permutation matrix $\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Write down $\boldsymbol{D}$ and $\boldsymbol{F}_3$ in terms of ω, and show your derivation.

4. Suppose you are given a general-purpose subroutine for solving initial value problems for systems of first-order ODEs $y' = f(t, y)$, which supports integration using Euler's method, backward Euler's method, trapezoid method, and fourth-order Runge-Kutta method. Describe how you would use this software to solve the following problems, in terms of how to reformulate the problem, which method to choose, and how to select time steps.

   (a) Describe how you can use this software to solve the second-order initial value problem for ODE

   $$u'' = t + u + u'$$

   with initial conditions $u(0) = 0$ and $u'(0) = 1$.

   (b) Describe how you can use this software to solve the heat equation

   $$u_t = cu_{xx},\ 0 \le x \le 1,\ t \ge 0,$$

   with initial condition

   $$u(0, x) = g(x),\ 0 \le x \le 1,$$

   and boundary conditions

   $$u(t, 0) = 0,\ u(t, 1) = 0,\ t \ge 0.$$

5. Quadrature rule.

   (a) Derive a two-point Gaussian quadrature rule for the interval $[0, 1]$. Give the equation for the coefficients and weights. (You do not need to solve the resulting equation.)

   (b) What is the degree of the resulting rule?

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(\boldsymbol{x}) = 0.5x_1^2 + 2.5x_2^2$.

   (a) Show the first step of the steepest descent method for minimizing $f(x)$ starting from $x_1 = 1$ and $x_2 = 1$.

   (b) Show the first step of the Newton's method for minimizing $f(x)$ starting from $x_1 = 1$ and $x_2 = 1$.