AMS527: Numerical Analysis II

More on Boundary Value Problems

Xiangmin Jiao

SUNY Stony Brook

April 15, 2009

Summary on Boundary Value Problems

- General nonlinear BVPs

$$u'' = f(t, u, u'), \quad a \leq t \leq b$$

  with BC $u(a) = \alpha$ and $u(b) = \beta$
- Methods we have considered
  - Shooting methods
  - Finite difference methods
  - Collocation methods
  - Galerkin methods

Analysis of Accuracy of Finite Difference Methods

- Consider a linear BVP

$$-u'' + p(x)u' + q(x)u = f(x), \quad a < x < b$$

  with BC $u(a) = \alpha$ and $u(b) = \beta$.
- We assume $q(x) > 0$, so the minus sign in front of u'' is important.
- Questions:
  - What is the equation from finite difference approximation?
  - What is the order of accuracy?

Equations of Finite Difference Approximation

- Use centered difference approximation

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + \frac{1}{6}h^2 u'''(\xi_{x,h})$$
$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \frac{1}{12}h^2 u^{(4)}(\eta_{x,h})$$

- Then in general

$$-\left(1 + \frac{1}{2}p(x)h\right) u(x-h) + \left(2 + q(x)h^2\right) u(x) - \left(1 - \frac{1}{2}p(x)h\right) u(x+h)$$
$$= h^2 f(x) + R(x,h),$$

where the remainder is given by

$$R(x,h) = \frac{1}{12}h^4 u^{(4)}(\eta_{x,h}) + \frac{1}{6}p(x)h^4 u'''(\xi_{x,h}) = \mathcal{O}(h^4).$$

Equations of Finite Difference Approximation

- Let $t_0 = a$ and $t_n = b$. Let U_k denote the approximation to $u(t_k)$.
- Therefore, we have

$$
\begin{aligned}
&(2 + h^2 q(x_1))U_1 - (1 - \frac{1}{2}hp(x_1))U_2 \\
=&(1 + \frac{1}{2}hp(x_1))\alpha + h^2 f(x_1), \\
&-(1 + \frac{1}{2}hp(x_i))U_{i-1} + (2 + h^2 q(x_i))U_i - (1 - \frac{1}{2}hp(x_i))U_{i+1} \\
=&h^2 f(x_i), \\
&-(1 + \frac{1}{2}p(x_{n-1})h)U_{n-2} + (2 + q(x_n)h^2)U_{n-1} \\
=&h^2 f(x_{n-1}) - \beta(1 - \frac{1}{2}p(x_{n-1})h
\end{aligned}
$$

- This can be written as linear system $\boldsymbol{Ax} = \boldsymbol{b}$.

Order of Accuracy of Approximation

Theorem

Assume p and q are both continuous, with $|p(x)| \leq P_M$ and $0 < Q_m \leq q(x)$ for all $x \in [a, b]$, then there exists a constant $C > 0$, independent of h, such that

$$\max_{1 \leq i \leq n-1} |u(x_i) - U_i) \leq Ch^2(\|\boldsymbol{u}^{(4)}\|_\infty + \|\boldsymbol{u}^{(3)}\|_\infty).$$

- In other words, solution of BVP using centered difference approximation is second order accurate.
- The analysis is not obvious, because both the size of $\boldsymbol{A}$ and its entries depend on h

Sketch of Proof

- Define $e_i = u(x_i) - U_i$, with $e_0 = e_n = 0$. Then

$$\begin{aligned}
(2 + h^2 q(x_1))e_1 - (1 - \frac{1}{2}hp(x_1))e_2 &= R_1 \\
-(1 + \frac{1}{2}hp(x_i))e_{i-1} + (2 + h^2 q(x_i))e_i - (1 - \frac{1}{2}hp(x_i))e_{i+1} &= R_i, \\
-(1 + \frac{1}{2}p(x_{n-1})h)e_{n-2} + (2 + q(x_n)h^2)e_{n-1} &= R_{n-1}
\end{aligned}$$

where $2 \leq i \leq n-2$ in second equation, and $R_k = R(x_k, h)$

Sketch of Proof Cont'd

- Suppose $|e_J| \geq |e_i|$ for all i, and assume for simplicity $J \neq 1$ and $J \neq n-1$. Then

$$\begin{aligned}(2+h^2q(x_i))|e_i| &\leq \left|1+\frac{1}{2}hp(x_J)\right||e_{J-1}| + \left|1-\frac{1}{2}hp(x_J)\right||e_{J+1}| + |R_J \\ &\leq \left|1+\frac{1}{2}hp(x_J)\right||e_J| + \left|1-\frac{1}{2}hp(x_i)\right||e_J| + |R_J| \\ &= 2|e_J| + |R_J|.\end{aligned}$$

  Therefore, $h^2q(x_J))|e_J| \leq |R_J|$ and

$$|e_J| \leq \frac{1}{h^2Q_m}\max_{0\leq x\leq 1}|R(x,h)| \leq \frac{1}{6Q_m}h^2(\|\boldsymbol{u}^{(4)}\|_\infty + \|\boldsymbol{u}^{(3)}\|_\infty).$$

- This material is from book J. Epperson, *An Introduction to Numerical Methods and Analysis*, Revised Edition, Wiley, 2007.