1. Answer whether the following statements are true or false and give a brief explanation.

(a) A small residual \( \| f(x) \| \) guarantees an accurate solution of a system of nonlinear equations \( f(x) = 0 \).

(b) When minimizing a quadratic function \( f(x) \) whose Hessian matrix is positive definite, Newton’s method may not converge because it has only local convergence properties.

(c) When fitting a function to noisy data, the noise can usually be suppressed by using higher degree polynomial interpolation.

(d) For \( n \geq 2 \), there can be more than one quadrature rule that have \( n \) nodes and have a degree \( 2n - 1 \).

(e) Monte Carlo method is the most efficient way to perform numerical integration in dimensions \( \leq 4 \).

(f) When solving a two-point boundary value problem for ODEs, the shooting method is stable as long as the BVP is stable.

(g) The fast Fourier transform (FFT) algorithm can speed up DFT for a sequence of length \( n \) only if \( n \) is a power of 2.

(h) Multiplying two polynomials of degree \( n \) with all nonzero coefficients require at least \( O(n^2) \) arithmetic operations.

(i) For solving a time-dependent partial differential equation, a finite difference method that is both consistent and stable converges to the true solution as the step sizes in time and in space go to zero.

(j) A dense direct solver is particularly efficient in solving the linear system arising from finite difference approximations to the Poisson equation in higher dimensions.

2. Quadrature rule.

(a) Derive an open two-point Newton-Cotes quadrature for the interval \([a, b]\). What are the resulting nodes and weights?

(b) What is the degree of the resulting rule?

3. Newton’s method for solving a scalar nonlinear equation \( f(x) = 0 \) requires computation of the derivative of \( f \) at each iteration. Suppose that we instead replace the true derivative with a constant \( d \), that is, we use the iteration scheme

\[
x_{k+1} = x_k - f(x_k)/d.
\]

(a) Under what condition on the value of \( d \) will this scheme be locally convergent?

(b) What will be the convergence rate, in general?

(c) Is there any value for \( d \) that would still yield quadratic convergence?

4. Consider a scalar valued function \( f \). Its Laplacian is \( \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \).
(a) Construct a second-order finite difference approximation to the Laplacian $\Delta f$ for the interior points of a rectangular grid over a rectangular domain. Give a justification why the approximation is second order.

(b) Derive a second-order accurate finite difference approximation along the boundary of the rectangular grid.

5. Suppose you are given a general-purpose subroutine for solving initial value problems for systems of first-order ODEs $y' = f(t, y)$, which supports integration using Euler’s method, backward Euler’s method, trapezoid method, and fourth-order Runge-Kutta method. Describe how you would use this software to solve the following problems, in terms of how to reformulate the problem, which method to choose, and how to select time steps. Describe how you can use this software to solve the heat equation

$$u_t = cu_{xx}, \quad a \leq x \leq b, \quad t \geq 0,$$

with initial condition

$$u(0, x) = f(x), \quad a \leq x \leq b,$$

and boundary conditions

$$u(t, a) = 0, \quad u(t, b) = 0, \quad t \geq 0.$$

6. All entries in the factorization of Fourier matrix $F_4$ involves powers of $\omega = -i$, the fourth root of 1:

$$F_4 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_2 \\ F_2 \end{bmatrix} P_4^T,$$

where $P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Write down $D$ and $F_2$, and show your derivation.