AMS527: Numerical Analysis II
Linear Programming

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Linear Programming

- Linear programming has linear objective function and linear equality and inequality constraints
- Example: Maximize profit of combination of wheat and barley, but with limited budget of land, fertilizer, and insecticide. Let $x_1$ and $x_2$ be areas planted for wheat and barley, we have linear programming problem

$$\begin{align*}
\text{maximize} & \quad c_1 x_1 + c_2 x_2 & \{\text{maximize revenue}\} \\
0 & \leq x_1 + x_2 \leq L & \{\text{limit on area}\} \\
F_1 x_1 + F_2 x_2 & \leq F & \{\text{limit on fertilizer}\} \\
P_1 x_1 + P_2 x_2 & \leq P & \{\text{limit on insecticide}\} \\
x_1 & \geq 0, \ x_2 \geq 0 & \{\text{nonnegative land}\}
\end{align*}$$

- Linear programming is typically solved by *simplex methods* or *interior point methods*
Standard Form of Linear Programming

- Linear programming has many forms. A standard form (called *slack form*) is

  $$\min \quad c \mathbf{x} \quad \text{subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq 0$$

- Simplex method and interior-point method requires slack form

- Previous example can be converted into standard form

  minimize \((-c_1)x_1 + (-c_2)x_2\) \{maximize revenue\}
  \[x_1 + x_2 + x_3 = L\] \{limit on area\}
  \[F_1x_1 + F_2x_2 + x_4 = F\] \{limit on fertilizer\}
  \[P_1x_1 + P_2x_2 + x_5 = P\] \{limit on insecticide\}
  \[x_1, x_2, x_3, x_4, x_5 \geq 0\] \{nonnegativity\}

  Here, \(x_3\), \(x_4\), and \(x_5\) are called *slack variables*
Duality

- $m$ equations $Ax = b$ have $m$ corresponding Lagrange multipliers in $y$
- Primal problem

  Minimize $c^T x$ subject to $Ax = b$ and $x \geq 0$

- Dual problem

  Maximize $b^T y$ subject to $A^T y \leq c$

- Weak duality: $b^T y \leq c^T x$ for any feasible $x$ and $y$
  - because $b^T y = (Ax)^T y = x^T (A^T y) \leq x^T c = c^T x$

- Strong duality: If both feasible sets of primal and dual problems are nonempty, then $c^T x^* = b^T y^*$ at optimal $x^*$ and $y^*$
Simplex Methods

- Developed by George Dantzig in 1947
- Key observation: Feasible region is convex polytope in $\mathbb{R}^n$, and minimum must occur at one of its vertices
- Basic idea: Construct a feasible solution at a vertex of the polytope, walk along a path on the edges of the polytope to vertices with non-decreasing values of the objective function, until an optimum is reached
- Simplex method in the worst case can be slow, because number of corners is exponential with $m$ and $n$
- However, its average-case complexity is polynomial time, and in practice, best corner is often found in $2m$ steps
Interior Point Methods

- First proposed by Narendra Karmarkar in 1984
- In contrast to simplex methods, interior point methods move through the interior of the feasible region
- Barrier problem

\[
\text{minimize } \mathbf{c}^T \mathbf{x} - \theta (\log x_1 + \cdots + \log x_n) \quad \text{with } A \mathbf{x} = \mathbf{b}
\]

When any \( x_i \) touches zero, extra cost \(-\theta \log x_i\) blows up
- Barrier problem gives approximate problem for each \( \theta \). Its Lagrangian is

\[
\mathcal{L}(\mathbf{x}, \mathbf{y}, \theta) = \mathbf{c}^T \mathbf{x} - \theta \left( \sum \log x_i \right) - \mathbf{y}^T (A \mathbf{x} - \mathbf{b})
\]

- The derivatives \( \frac{\partial \mathcal{L}}{\partial x_j} = c_j - \frac{\theta}{x_j} - (A^T \mathbf{y})_j = 0 \), or \( x_j s_j = \theta \), where \( s = \mathbf{c} - A^T \mathbf{y} \)
Newton Step

- $n$ optimality equations $x_j s_j = \theta$ are nonlinear, and are solved iteratively using Newton’s method.

- To determine increment $\Delta x$, $\Delta y$, and $\Delta s$, we need to solve $(x_i + \Delta x_i)(s_i + \Delta s_i) = \theta$. It is typical to ignore second order term $\Delta x_i \Delta s_i$. Then linear equations become

\[
\mathbf{A} \Delta \mathbf{x} = 0 \\
\mathbf{A}^T \Delta \mathbf{y} + \Delta \mathbf{s} = 0 \\
s_j \Delta x_j + x_j \Delta s_j = \theta - x_j s_j.
\]

- The iteration has quadratic convergence for each $\theta$, and $\theta$ approaches zero.

Example

Minimize $c^T x = 5x_1 + 3x_2 + 8x_3$ with $x_i \geq 0$ and $Ax = x_1 + x_2 + 2x_3 = 4$.

1. Barrier Lagrangian is
   \[\mathcal{L} = (5x_1 + 3x_2 + 8x_3) + \theta(\log x_1 + \log x_2 + \log x_3) - y(x_1 + x_2 + 2x_3 - 4)\]

2. Optimality equation gives us:
   \[s = c - A^T y\]
   \[\partial \mathcal{L} / \partial x_i = 0\]
   \[\partial \mathcal{L} / \partial y = 0\]
   \[x_1s_1 = x_2s_2 = x_3s_3 = \theta\]
   \[x_1 + x_2 + 2x_3 = 4\]

3. Start from an interior point $x_1 = x_2 = x_3 = 1$, $y = 1$, and $s = (3, 1, 4)$. From $A\Delta x = 0$ and $x_j s_j + s_j \Delta x_j + x_j \Delta s_j = \theta$, we obtain equations
   \[3\Delta x_1 - 1\Delta y = \theta - 3\]
   \[1\Delta x_2 - 1\Delta y = \theta - 1\]
   \[4\Delta x_3 - 2\Delta y = \theta - 4\]
   \[\Delta x_1 + \Delta x_2 + 2\Delta x_3 = 0\]

4. Given $\theta = 4/3$, we then obtain $x_{new} = (2/3, 2, 2/3)$ and $y_{new} = 8/3$, whereas $x^* = (0, 4, 0)$ and $y^* = 3$. 