Consider the problem

\[ \min_{x,y} f(x,y) = x^2 + y^2 \]
subject to
\[ g(x,y) = y^2 - (x - 1)^3 = 0. \]
(a) First try to solve this problem using the method of Lagrange multipliers. Explain why this method fails for this problem.
(b) Next try the penalty method given in Section 6.7.2 to solve this problem, i.e. solve

\[ \min_{x,y} f(x,y) + \frac{1}{2} \rho g(x,y)^2. \]

Derive an expression for the solution to the latter problem as a function of \( \rho \) and then take the limit as \( \rho \to \infty \).

2. (10 points) Exercise 7.4 on page 335 of Heath book
How many multiplications are required to evaluate a polynomial \( p(t) \) of degree \( n-1 \) at a given point \( t \)
(a) Represented in the monomial basis?
(b) Represented in the Lagrange basis?
(c) Represented in the Newton basis?

3. (10 points) Exercise 7.9 on page 336 of Heath book
Use the Lagrange interpolation to derive the formulas given in Section 5.5.5 for inverse quadratic interpolation.

Prove that the formula using divided differences given in Section 7.3.3,

\[ x_j = f[t_1, t_2, \ldots, t_j], \]

indeed gives the coefficient of the \( j \)th basis function in the Newton polynomial interpolant.
Hint: Prove by induction. When interpolating points \( (t_1, y_1), \ldots, (t_j, y_j) \), observe that the interpolant \( p_j(t) \) can be written as

\[ p_j(t) = p_{j-1}(t) + \frac{t - t_1}{t_j - t_1} (r(t) - p_{j-1}(t)), \]

where \( r(t) \) interpolates \( (t_2, y_2), \ldots, (t_j, y_j) \). Then use induction hypothesis on \( p_{j-1}(t) \) and \( r(t) \).
   (a) Verify that the Chebyshev polynomials of the first kind, as defined in Section 7.3.4, satisfy the
   three-term recurrence given there.
   (b) Verify that the first six Chebyshev polynomials are as listed in Section 7.3.4.
   (c) Verify that the expression for the roots and extrema of $T_k$ given in Section 7.3.4 are correct.

   Compute both polynomial and cubic spline interpolants to Runge’s function, $f(t) = 1/(1 + 25t^2)$, using
   both $n = 11$ and 21 equally spaced points on the interval $[-1, 1]$. Compare your results graphically by
   plotting both interpolants and the original function for each value of $n$.

   The gamma function is defined by
   \[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} dt, \quad x > 0. \]
   For an integer argument $n$, the gamma function has the value
   \[ \Gamma(n) = (n - 1)!, \]
   so interpolating the data points
   \[
   \begin{array}{cccc}
   t & 1 & 2 & 3 \\
   y & 1 & 2 & 6 \\
   \end{array}
   \] should yield an approximation to the gamma function over the given range.
   (a) Compute the polynomial of degree four that interpolates these five data points. Plot the resulting
   polynomial as well as the corresponding values given by the built-in gamma function over the domain
   $[1, 5]$.
   (b) Use a cubic spline routine to interpolate the same data and again plot the resulting curve along
   with the built-in gamma function.
   (c) Which of the two interpolants is more accurate over most of the domain?
   (d) Which of the two interpolants is more accurate between 1 and 2?