AMS527: Numerical Analysis II
Supplementary Material on Finite Difference Methods for Two-Point Boundary-Value Problems

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Consider simple second-order ODE

\[ u''(x) = f(x) \text{ for } 0 < x < 1, \]

with some given boundary conditions \( u(0) = \alpha \) and \( u(1) = \beta \).

FDM computes grid functions of \( U_0, U_1, \ldots, U_m, U_{m+1} \), where \( U_j \) is approximation to solution \( u(x_j) \) at \( j \)th grid point \( x_j = jh \) and \( h = 1/(m+1) \) is mesh width.

Replace \( u''(x) \) by finite difference operator (such as centered difference approximation)

\[ D^2 U_j = \frac{1}{h^2} (U_{j-1} - 2U_j + U_{j+1}), \]

we obtain a set of algebraic equations

\[ \frac{1}{h^2} (U_{j-1} - 2U_j + U_{j+1}) = f(x_j) \text{ for } j = 1, 2, \ldots, m, \]

or

\[ AU = F. \]
Linear System from Simple FDM

\[
\frac{1}{h^2} \begin{bmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
& 1 & -2 & 1 & \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -2 & 1 \\
& & & & 1 & -2 \\
\end{bmatrix} \begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
\vdots \\
U_{m-1} \\
U_m \\
\end{bmatrix} = \begin{bmatrix}
f(x_1) - \alpha/h^2 \\
f(x_2) \\
f(x_3) \\
\vdots \\
f(x_{m-1}) \\
f(x_m) - \beta/h^2 \\
\end{bmatrix}
\]
Let $\hat{U} = [u(x_1), u(x_2), \ldots, u(x_m)]$ be vector of true values. Error vector is $E = U - \hat{U}$, composed of errors at each grid point. Error norms typically used are

\[
\|E\|_\infty = \max_{1 \leq j \leq m} |E_j|
\]

\[
\|E\|_1 = h \sum_{j=1}^{m} |E_j|
\]

\[
\|E\|_2 = \sqrt{h \sum_{j=1}^{m} |E_j|^2}.
\]

$E = U - \hat{U}$ is referred to as global error.
Local Truncation Error (LTE)

- Local truncation error (LTE), denoted by \( \tau_j \), i.e.,
  \[ \tau_j = D^2(u) - f(x_j) \] for \( j = 1, 2, \ldots, m \)
- For centered difference approximation,
  \[
  \tau_j = \frac{1}{h^2} (u_{j-1} - 2u_j + u_{j+1}) - f(x_j)
  = \left[ u''(x_j) + \frac{1}{12} h^2 u'''(x_j) + O(h^4) \right] - f(x_j)
  = \frac{1}{12} h^2 u'''(x_j) + O(h^4).
  
- Assume \( u''' \) is bounded, then \( \tau_j = O(h^2) \) as \( h \to 0 \)
- Let \( \tau \) be vector composed of \( \tau_j \), then \( \tau = A\hat{U} - F \), or \( A\hat{U} = \tau + F \).
- Therefore,
  \[ AE = A(U - \hat{U}) = -\tau. \]
Stability

Suppose a finite difference method for a linear BVP gives a sequence of matrix equations of form

\[ A^h U^h = F^h, \]

where \( h \) is the mesh width, and the superscript denotes solution on grid with mesh grading \( h \).

We say the method is stable if \((A^h)^{-1}\) exists for all \( h \) sufficiently small (say \( h < h_0 \)) and if there is a constant \( C \), independent of \( h \), such that

\[ \| (A^h)^{-1} \| \leq C \quad \text{for all} \quad h < h_0. \]
Consistency

We say that a method is consistent with differential equation and boundary conditions if

$$
\|\tau^h\| \to 0 \text{ as } h \to 0.
$$

Typically, \(\|\tau^h\| = O(h^p)\) for some integer \(p > 0\), and so it is consistent.
A method is said to be *convergent* if $\|E^h\| \to 0$ as $h \to 0$.

In general,

$$\text{consistency} + \text{stability} \implies \text{convergence}.$$ 

This is because

$$\|E^h\| \leq \|(A^h)^{-1}\| \|\tau^h\| \leq C \|\tau^h\| \to 0 \text{ as } h \to 0.$$ 

This is known as *fundamental theorem of finite difference methods*.

This is analogous to *Lax equivalence theorem* for finite difference methods for the numerical solution of partial differential equations: “For a consistent finite difference method for a well-posed linear initial value problem, the method is convergent if and only if it is stable.”
Challenge in proof of convergence often lies in verification of “stability” (instead of consistency).

In 2-norm, for symmetric matrices

\[ \|A\|_2 = \rho(A) = \max_{1 \leq \rho \leq m} |\lambda_{\rho}| \]

and

\[ \|A^{-1}\|_2 = \rho(A^{-1}) = \left( \min_{1 \leq \rho \leq m} |\lambda_{\rho}| \right)^{-1} . \]

Therefore, one needs to show eigenvalues of \( A \) is bounded away from zero as \( h \to 0 \).
The $m$ eigenvalues of $A$ are

$$\lambda_p = \frac{2}{h^2} (\cos(p\pi h) - 1)$$

for $p = 1, 2, \ldots, m$,

and their corresponding eigenvector $u^p$ has $j$th component $u^p_j = \sin(p\pi jh)$.

Smallest eigenvalue of $A$ (in magnitude) is

$$\lambda_1 = \frac{2}{h^2} (\cos(\pi h) - 1)$$

$$= \frac{2}{h^2} \left( -\frac{1}{2} \pi^2 h^2 + \frac{1}{24} \pi^4 h^4 + O(h^6) \right)$$

$$= -\pi^2 + O(h^2),$$

which is bounded away from 0 as $h \to 0$

Therefore, $\|E^h\|_2 \leq \frac{1}{\pi^2} \|\mathbf{\tau}^h\|_2$, so $\|E\|_2 = O(h^2)$

It can also be shown that $\|E\|_\infty = O(h^2)$, but it is more involved
Consider simple second-order ODE

\[ u''(x) = f(x) \quad \text{for } 0 < x < 1, \]

with boundary conditions

\[ u'(0) = \sigma, \quad u(1) = \beta \]

Discretize interval \([0, 1]\) by introducing \(m\) interior points such that

\[ 0 = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m < x_{m+1} = 1 \]

First Attempt: Approximate \(u'(0) = 0\) using one-sided derivative

\[ \frac{U_1 - U_0}{h} = \sigma \]
Then equation becomes

\[
\begin{bmatrix}
-\frac{1}{h^2} & h & 1 \\
1 & -2 & 1 \\
1 & -2 & 1 \\
\vdots & \vdots & \vdots \\
1 & -2 & 1 \\
0 & h^2
\end{bmatrix}
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_m \\
U_{m+1}
\end{bmatrix}
= \begin{bmatrix}
\sigma \\
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_m) \\
\beta
\end{bmatrix}
\]

Local truncation error is $O(h)$, because

\[
\tau_0 = \frac{1}{h^2}(hu(x_1) - hu(x_0)) - \sigma
\]

\[
= u'(x_0) + \frac{1}{2}hu''(x_0) + O(h^2) - \sigma
\]

\[
= \frac{1}{2}hu''(x_0) + O(h^2),
\]

This translates into a global error of $O(h)$.
Use centered approximation to \( u'(0) = \sigma \) by introducing \( U_{-1} \)

\[
\frac{1}{h^2}(U_{-1} - 2U_0 + U_1) = f(x_0)
\]

\[
\frac{1}{2h}(U_1 - U_{-1}) = \sigma
\]

Eliminate \( U_{-1} \) from the two equations and we obtain

\[
\frac{1}{h}(-U_0 + U_1) = \sigma + \frac{h}{2}f(x_0)
\]

We then have

\[
\frac{1}{h^2} \begin{bmatrix} -h & h \\ 1 & -2 & 1 \\ 1 & -2 & 1 \\ \vdots & \vdots & \vdots \\ 1 & -2 & 1 \\ 0 & h^2 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma + \frac{h}{2}f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}
\]
Second Second-Order Accurate Approach

- Use second-order one-sided approximation to \( u'(0) = \sigma \)

\[
\frac{1}{h} \left( \frac{3}{2} U_0 - 2 U_1 + \frac{1}{2} U_2 \right) = \sigma
\]

- We then have

\[
\frac{1}{h^2} \begin{bmatrix}
\frac{3}{2} h & -2h & \frac{1}{2} h \\
1 & -2 & 1 \\
1 & -2 & 1 \\
\vdots & \vdots & \vdots \\
1 & -2 & 1 \\
0 & h^2
\end{bmatrix}
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
\vdots \\
U_m \\
U_{m+1}
\end{bmatrix}
= \begin{bmatrix}
\sigma \\
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_m) \\
\beta
\end{bmatrix}
\]

- This approach increases bandwidth of matrix, but is more general and easier to generalize to higher order accuracy and to nonuniform grids