Runge-Kutta Methods

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General Form of Runge-Kutta Methods

\[ u_{n+1} = u_n + hF(t_n, u_n, h; f), \quad n \geq 0, \]

where

\[ F(t_n, u_n, h; f) = \sum_{i=1}^{s} b_i K_i, \]

and

\[ K_i = f(t_n + c_i h, u_n + h \sum_{j=1}^{s} a_{ij} K_j), \quad i = 1, 2, \ldots, s \]
Butcher Array

\[
\begin{array}{c|cccc}
  c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\
  c_2 & a_{21} & a_{22} & & a_{2s} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  c_s & a_{s1} & a_{s2} & \ldots & a_{ss} \\
\hline
  b_1 & b_2 & \ldots & b_s
\end{array}
\]

where \( A = (a_{ij}) \in \mathbb{R}^{s \times s} \), \( b = (b_1, \ldots, b_s)^T \in \mathbb{R}^s \), and \( c = (c_1, \ldots, c_s)^T \in \mathbb{R}^s \)

- If \( A \) is strictly lower triangular (i.e., \( a_{ij} = 0 \) for \( j \geq i \)), then each \( K_i \) can be computed explicitly in terms of \( K_1, \ldots, K_{i-1} \), so method is *explicit*
- Otherwise, method is *implicit*
- RK method is *semi-implicit* if i.e., \( a_{ij} = 0 \) for \( j > i \)


- In general,
  \[ c_i = \sum_{j=1}^{s} a_{ij}, \quad i = 1, \ldots, s \]

  and
  \[ \sum_{i=1}^{s} b_i = 1 \]

- Let (local) truncation error \( \tau_{n+1}(h) \) to be defined as
  \[ h\tau_{n+1}(h) = y_{n+1} - y_n - hF(t_n, y_n, h; f), \]
  where \( y(t) \) is the exact solution to the Cauchy problem

- Method is consistent if \( \tau_{n+1}(h) = \max_n |\tau_n(h)| \to 0 \) as \( h \to 0 \)

- Method is order \( p (\geq 1) \) with respect to \( h \) if \( \tau(h) = O(h^p) \) as \( h \to 0 \)
The order of an $s$-stage explicit Runge-Kutta method cannot be greater than $s$. Also, there do not exist $s$-stage Runge-Kutta method with order $s$ if $s \geq 5$.

<table>
<thead>
<tr>
<th>order</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{\min}$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>
Explicit fourth-order RK method is provided by

\[ u_{k+1} = u_k + \frac{h}{6} (K_1 + 2K_2 + 2K_3 + K_4), \]

where

\[ K_1 = f_n \]
\[ K_2 = f(t_n + \frac{h}{2}, u_n + \frac{h}{2}K_1) \]
\[ K_3 = f(t_n + \frac{h}{2}, u_n + \frac{h}{2}K_2) \]
\[ K_4 = f(t_{n+1}, u_n + hK_3) \]
Derivation of Explicit RK Method

- Consider 2-stage explicit RK method

\[ u_{n+1} = y_n + hF(t_n, y_n, h; f) = y_n + h(b_1 K_1 + b_2 K_2), \]

where

\[ K_1 = f(t_n, y_n), \quad K_2 = f(t_n + h c_2, y_n + h c_2 K_1) \]

- Expand \( K_2 \) in a Taylor series

\[ K_2 = f_n + h c_2(f_{n,t} + K_1 f_{n,y}) + O(h^2) \]

where \( f_{n,y} \) denotes the partial derivative of \( f \) with respect to \( y \) evaluated at \((t_n, y_n)\)

- Then

\[ u_{n+1} = y_n + h f_n (b_1 + b_2) + h^2 c_2 b_2 (f_{n,t} + f_n f_{n,y}) + O(h^3) \]
Derivation of Explicit RK Method Cont’d

- Perform same expansion on exact solution, we find

\[ y_{n+1} = y_n + hy_n' + \frac{h^2}{2} y_n'' + O(h^3) \]

\[ = y_n + h(b_1 + b_2)f_n + h^2 c_2 b_2 (f_{n,t} + f_n f_{n,y}) + O(h^3) \]

- Forcing coefficients in two expansions to agree up to higher-order terms, we obtain

\[ b_1 + b_2 = 1 \]

\[ c_2 b_2 = \frac{1}{2} \]

- Therefore, there are infinitely many 2-stage explicit RK methods with second-order accuracy. Heun’s method is one of them.
Region of Stability for $s$-stage Explicit RK Methods

- Regions of stability for $s$-stage explicit RK, with $s = 1, \ldots, 4$.
- Only shows portion $\text{Im}(h\lambda) \geq 0$; symmetric about real axis.
- Stability region increases as number of stages increases.
Implicit RK Methods

- Implicit RK methods can be derived based on integral formulation
  \[ y(t) - y_0 = \int_{t_0}^{t} f(\tau, y(\tau)) d\tau \]

- Apply quadrature formula with \( s \) nodes in \((t_n, t_{n+1})\), we get
  \[ \int_{t_n}^{t_{n+1}} f(\tau, y(\tau)) d\tau \approx h \sum_{j=1}^{s} b_j f(t_n + c_j h), \]

  where \( b_j \) correspond to weights and \( t_n + c_j h \) correspond to quadrature nodes
Examples of Implicit RK Methods

- Examples include Gauss-Legendre RK methods, which use Gauss-Legendre quadrature points, with order up to $2s$
  
  - 1-stage (implicit midpoint method) \[
  \begin{array}{c|cc}
  \frac{1}{2} & \frac{1}{2} & 1 \\
  \hline
  \frac{3-\sqrt{3}}{6} & \frac{1}{4} & \frac{3-2\sqrt{3}}{12} \\
  \frac{3+\sqrt{3}}{6} & \frac{3+2\sqrt{3}}{12} & \frac{1}{4} \\
  \frac{1}{2} & \frac{1}{2} \\
  \end{array}
  \]
  is order 2

  - 2-stage given by \[
  \begin{array}{c|cc}
  \frac{1}{2} & \frac{1}{2} & 1 \\
  \hline
  \frac{3-\sqrt{3}}{6} & \frac{1}{4} & \frac{3-2\sqrt{3}}{12} \\
  \frac{3+\sqrt{3}}{6} & \frac{3+2\sqrt{3}}{12} & \frac{1}{4} \\
  \frac{1}{2} & \frac{1}{2} \\
  \end{array}
  \]
  is order 4

- Gauss-Radau methods use Gauss-Radau quadrature points, which include one of two end points, with order up to $2s - 1$

- Gauss-Lobatto methods use Gauss-Lobatto quadrature points, which include two end points, with order up to $2s - 2$