AMS 529/691 Homework 1

Due: Monday 09/18 in class.

1. (10 points) Use three hat functions, with \( h = \frac{1}{4} \), to solve \(-u'' = 2\) with \( u(0) = u(1) = 0 \). Verify that the approximation \( U \) matches \( u = x - x^2 \) at the nodes.

2. (10 points) Replace the boundary conditions in Problem 1 with \( u(0) = 1 \) and \( u'(1) = 1 \). Modify the linear system for problem 1.

3. (15 points) Given a set of trial functions \( \phi_1, \phi_2, \ldots, \phi_n \) over \( \Omega \in \mathbb{R}^2 \) with \( \sum_{i=1}^{n} \phi_i = 1 \), the mass matrix \( M \) is defined as

\[
m_{ij} = \int_{\Omega} \phi_i \phi_j \, dx \, dy, \quad i, j = 1, 2, \ldots, n.
\]

Show that \( M \) is symmetric and positive definite.

4. (15 points) Consider a triangle \( \tau \) with points \( x_i = (x_i, y_i), i = 1, 2, 3 \). Let \( \phi_i(x, y) = a_i + b_i x + c_i y, \) \( i = 1, 2, 3 \), denote the Lagrange linear polynomials over the triangle with \( \phi_i(x_j, y_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \).

(a) (5 points) Show that

\[
\begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
\end{bmatrix}
= \begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \\
\end{bmatrix}^{-1}
.
\]

(b) (5 points) Derive the formula for the \( a_i, b_i \) and \( c_i \).

(c) (5 points) Derive the formula for \( \nabla \phi_i \).

Note: For parts (b) and (c), you can use a computer algebra system, such as MAPLE, Mathematica, MATLAB Symbolic Toolbox, Octave’s Symbolic package, or Python’s sympy module.

5. (20 points) The classical way to derive the 2-D hat functions over a triangle \( \tau \) is to define a linear mapping \( g(\xi) \), such that

\[
x_i = g(\xi_i),
\]

where \( \xi_1 = (0, 0), \xi_2 = (1, 0), \) and \( \xi_3 = (0, 1) \), and then define the shape functions \( N_i(\xi) \) such that

\[
g(\xi) = \sum_{i=1}^{3} x_i N_i(\xi),
\]

where \( x_i = (x_i, y_i), i = 1, 2, 3 \) are the nodes of the triangle. For any smooth function \( u(x) \), we can approximate \( u \) within the triangle as

\[
u(x) \approx \sum_{i=1}^{3} u(x_i) \phi_i(x) = \sum_{i=1}^{3} u(x_i) N_i(g^{-1}(x)),
\]

i.e., \( \phi_i(x) = N_i(g^{-1}(x)) \) and \( N_i(\xi) = \phi_i(g(\xi)) \).

(a) (5 points) Show that \( N_1(\xi) = 1 - \xi - \eta, N_2(\xi) = \xi, \) and \( N_3(\xi) = \eta. \)
(b) (5 points) The Jacobean matrix $J$ for the mapping $g(\xi)$ is $J = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial \bar{z}}{\partial \xi} & \frac{\partial \bar{z}}{\partial \eta} \end{bmatrix}$. Show that

$$J = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}.$$ 

(c) (5 points) Show that $\nabla \phi_i(x) = J^{-1} \nabla N_i(\xi)$.

(d) (5 points) Express $\int \nabla \phi_i(x) \cdot \nabla \phi_j(x) dxdy$ in terms of $J$ and $\nabla N_i(\xi)$ for $i, j = 1, 2, 3$.

6. (30 points) Consider the Poisson equation

$$-\Delta u = 2$$

over $\Omega = [0, 1] \times [0, 1]$ with $u = 0$ on $\partial \Omega$. Use linear FEM in FEniCS to solve the equation. Submit your code and the plot of the solution.