AMS 691/529: Finite Element Methods: Theory and Computations
Lecture 8: Error Analysis of FEM for Poisson Equations; Superconvergence of Nodal Solutions

Xiangmin Jiao

SUNY Stony Brook
Outline

1. Error Analysis in Energy Norm

2. $L^2$ Error and Aubin-Nitsche Duality Argument

3. Superconvergence of FEM
Energy Norm and Properties

We first consider classical analysis in energy norm (or semi-norm).

**Definition 1**

*Energy norm* of Poisson eq. \(- \nabla \cdot (c \nabla u) = f\) over \(\Omega\) with Dirichlet BC is

\[
\|u\|_E := \sqrt{\int_{\Omega} c \nabla u \cdot \nabla u \, dx},
\]

and its corresponding *inner product* is \(\langle u, v \rangle_E := \int_{\Omega} c \nabla u \cdot \nabla v \, dx\).

- Triangle inequality:

\[
\|u + v\|_E \leq \|u\|_E + \|v\|_E
\]

- Cauchy-Schwarz inequality:

\[
\langle u, v \rangle_E \leq \langle \|u\|, \|v\| \rangle_E \leq \|u\|_E \cdot \|v\|_E
\]
Orthogonality of Residual

**Proposition 1**

If $u_h(x)$ is solution of Galerkin method with trial space $\mathcal{V}$ for Poisson equation over $\Omega$ with Dirichlet BC, then

$$\langle u_h - u, v \rangle_E = 0$$

for all $v \in \mathcal{V}$.

**Proof.**

$$\langle u_h - u, v \rangle_E = \langle u_h, v \rangle_E - \langle u, v \rangle_E.$$  \hfill (2)

From variational form, $\langle u_h, v \rangle_E = \int_{\Omega} c \nabla u \cdot \nabla v \, dx = \int_{\Omega} fv \, dx$.

From weighted residual, $\langle u, v \rangle_E = \int_{\Omega} c \nabla u \cdot \nabla v \, dx = \int_{\Omega} fv \, dx$.

Therefore, RHS of (2) is equal to 0.
Minimization of Energy Norm

**Lemma 1**

If \( u_h(x) \) is solution of Galerkin method in \( \mathcal{V} \) for Poisson equation, then

\[
\| u_h - u \|_E \leq \| v - u \|_E
\]

for all \( v \in \mathcal{V} \).

**Proof.**

Let \( u_h - u = u_h - v + w \) for any \( v \in \mathcal{V} \), where \( w = v - u \in \mathcal{V} \). Then

\[
\| u_h - u \|_E^2 = \langle u_h - u, u_h - v + w \rangle_E \quad \{ \text{substitution} \}
\]

\[
= \langle u_h - u, u_h - v \rangle_E \quad \{ \text{bilinearity & orthogonality} \}
\]

\[
\leq \| u_h - u \|_E \| u_h - v \|_E \quad \{ \text{Cauchy-Schwarz} \}
\]

Divide both sides by \( \| u_h - u \|_E \) and we have (3).
Accuracy of Gradient of Linear Interpolation

For vector- or tensor-valued $\mathbf{\sigma}$ over $\Omega$, define $\|\mathbf{\sigma}\|_{L^p(\Omega)} := \|\|\mathbf{\sigma}\|_2\|_{L^p(\Omega)}$.

Lemma 2

Let $u_I = \sum_{j=1}^{n} u(x_j) \phi_j(x)$ with piecewise linear Lagrange basis. Then

$$\|\nabla u_I - \nabla u\|_{L^p} \leq \|\cot \theta_{\text{min}} D^2 u\|_{L^p} O(h),$$

(4)

for $1 \leq p \leq \infty$, where $\theta_{\text{min}}$ is the minimum angles within all faces of elements, $h$ is max edge length, and $D^2 g = \|\nabla^2 g\|_2$.

- From properties of FEM interpolation, $\exists C$, s.t.

$$\|\nabla u_I - \nabla u\|_{L^p} \leq C \|D^2 u\|_{L^p} O(h),$$

similar to proof of $\|u_I - u\|_{L^p}$ in analysis of $L^2$ projection.
- $\cot \theta_{\text{min}}$ corresponds to min-angle condition, which is not tight.
Accuracy of Galerkin Method in Energy Norm

Preceding lemmas imply following result of linear FEM:

**Theorem 1**

If $u_h(x)$ is solution of Galerkin method of Poisson equation over $\Omega$ with Dirichlet BC with piecewise linear basis function, then

$$
\| u_h - u \|_E \leq \| \cot \theta_{\text{min}} D^2 u \mathcal{O}(\ell) \|_{L^2} \leq \| \cot \theta_{\text{min}} \|_{L^2} \| D^2 u \|_{L^2} \mathcal{O}(h),
$$

where $\ell, h = \| \ell \|_{\infty}$, and $\theta_{\text{min}}$ is minimum angle with all faces of elements.

- In general, degree-$k$ basis gives $O(h^k)$ accuracy in energy norm if
  - $u$ is $C^k$ continuous
  - mapping from parametric to real space is affine transformation
  - $\cot \theta_{\text{min}}$ is bounded and $\partial \Omega$ is piecewise smooth

- Analysis in energy norm also holds for Neumann BC and Robin BC

- $L^2$ norm is $\leq O(h)$ due to Poincaré’s inequality, for $1 \leq p < \infty$,

$$
\| u \|_{L^p(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)}.
$$

Its generalization is to higher-order derivatives is Friedrichs’s inequality
Outline

1. Error Analysis in Energy Norm

2. $L^2$ Error and Aubin-Nitsche Duality Argument

3. Superconvergence of FEM
Error Bounds in $L^2$ Norm

**Proposition 2**

Let $u_h(x)$ be solution of linear Galerkin method for Poisson equation

\[-\nabla \cdot (c \nabla u) = f\]

over $\Omega$ with Dirichlet BC. Then,

$$\|u_h - u\|_{L^2} \leq C \|D^2 u\|_{L^2} O(h^2), \quad (5)$$

where $h$ is maximum edge length.

- $C$ depends on angles
- In classical analysis, error bound is derived using *Nitsche’s trick* (a.k.a. *Aubin-Nitsche duality argument*)
Aubin-Nitsche Duality Argument

- To bound $\|u_h - u\|_{L^2}$, assume $w$ is solution of dual problem
  
  $$ - \nabla \cdot (c \nabla w) = u_h - u $$

  over $\Omega$ with homogeneous Dirichlet BC for $w$ on $\partial \Omega$

- For any $v \in V$,

  $$ \|u_h - u\|^2_{L^2} = \int_{\Omega} - \nabla \cdot (c \nabla w)(u_h - u) \, dx $$

  $$ = \int_{\Omega} c \nabla w \cdot \nabla (u_h - u) \, dx $$

  $$ = \int_{\Omega} c \nabla (w - v) \cdot \nabla (u_h - u) \, dx $$

  $$ \leq \|w - v\|_E \|u_h - u\|_E $$

  \{orthogonality\}

- In general,

  $$ \|\nabla \cdot (c \nabla w)\|_{L^2} \leq O(1) \|D^2 w\|_{L^2} $$

  This is known as $H^2$ regularity (or ellipticity) assumption
Aubin-Nitsche Duality Argument Cont’d

- Let \( w_h \in \mathcal{V} \) be solution of dual problem. Then,

\[
\| w_h - w \|_E \leq \| D^2 w \|_{L^2 O(h)},
\]

(7)

- Assuming \( \| u_h - u \|_{L^2} \neq 0 \),

\[
\| u_h - u \|_{L^2} \leq \frac{\| w_h - w \|_E}{\| u_h - u \|_{L^2}} \| u_h - u \|_E
\]

\[
\leq \frac{\| D^2 u \|_{L^2 O(h)}}{\| \nabla \cdot (c \nabla w) \|_{L^2}} \| u_h - u \|_E
\]

\[
\leq O(h) \| u_h - u \|_E \quad \text{\{H}^2 \text{ regularity\}}
\]

\[
\leq \| D^2 u \|_{L^2 O(h^2)} \quad \text{\{bound on} \ \| \cdot \|_E \text{\}}
\]
Remarks on Duality Argument

- Does the duality argument work with high-order degree-$k$ FEM?

- Yes. It bounds $L^2$ norm error to $O(h^k)$.

- However, Neumann BC and curved boundaries can lead to complications in its analysis.

- More importantly, nodal values may be more accurate (even exact), but the classical Aubin-Nitsche duality argument cannot explain it.

- This is because energy norm and $L^2$ norm have intrinsic interpolation errors, which are $O(h^k)$ and $O(h^k+1)$, respectively.

- In other words, errors in classical Aubin-Nitsche duality argument are limited by interpolation errors.
Remarks on Duality Argument

- Does the duality argument work with high-order degree-$k$ FEM?
- Yes. It bounds $L^2$ norm error to $O(h^{k+1})$
- However, Neumann BC and curved boundaries can lead to complications in its analysis
Remarks on Duality Argument

- Does the duality argument work with high-order degree-$k$ FEM?
  - Yes. It bounds $L^2$ norm error to $O(h^{k+1})$
- However, Neumann BC and curved boundaries can lead to complications in its analysis
- More importantly, nodal values may be more accurate (even exact), but the classical Aubin-Nitsche duality argument cannot explain it
  - This is because energy norm and $L^2$ norm have intrinsic interpolation errors, which are $O(h^k)$ and $O(h^{k+1})$, respectively
  - In other words, errors in classical Aubin-Nitsche duality argument are limited by interpolation errors
Outline

1. Error Analysis in Energy Norm

2. $L^2$ Error and Aubin-Nitsche Duality Argument

3. Superconvergence of FEM
Superconvergence of Nodal Solutions

When solving Poisson equation on uniform meshes, if load vector is integrated exactly, nodal error is sometimes higher-order than $O(h^{k+1})$

- **Numerical experimentation of superconvergence**
- **Key observations**
  - In 1-D, Poisson solver is $\infty$th order (up to machine precision)
  - In 2-D and 3-D, Poisson solver may achieve $O(h^{k+2})$ accuracy at nodes for even $k$ on uniform meshes
- **Bottom line**
  - Do not use $L^2$ norm (or energy norm) to estimate errors, since interpolation errors may dominate
  - $l^p$ norm of nodal errors, $\|u\|_p/\sqrt{n}$, is more appropriate as error metric
- **Related superconvergence in finite differences**
  - 9-node finite-differences for Laplacian operator achieves $O(h^4)$
  - Its equation is same as center node of bi-quadratic (9-node rectangle) FEM with Dirichlet BC
Proof of $\infty$-Order Accuracy in 1D

- $\infty$-order accuracy in 1-D was proven by Dupont and Douglas in 1970s
- Let $G_0(x)$ be *Green’s function* corresponding to point load $\delta(x - x_0)$,
  \[
  G_0''(x) = \delta(x - x_0)
  \]
- $G_0(x)$ is piecewise linear with change of slope at $x_0$
- If $x_0$ is a node, $G_0(x)$ is in test space. FEM solution $u_h$ satisfies
  \[
  u_h(x_0) - u(x_0) = \langle \delta(x - x_0), u_h(x) - u(x) \rangle \\
  = a(G_0(x), u_h(x) - u(x)) \\
  = a(u_h(x) - u(x), G_0(x)) \\
  = 0
  \]
- The Dupont-Douglas argument does not generalize to 2D or 3D, since Green’s function at a node is no longer in test space