4.2 Mixed Finite Element Approximation

This section is concerned with the mixed finite element approximation of the Stokes problem. Using the theoretical results of §2.4.2, we show that the discrete spaces for the velocity and the pressure must satisfy an inf-sup compatibility condition to ensure the well-posedness of the discrete problem.

4.2.1 The compatibility condition

Let $X_h \subset [H^1_0(\Omega)]^d$ and $M_h \subset L^2_{f=0}(\Omega)$ be finite-dimensional spaces. Using the notation of §4.1.2, consider the discrete problem:

\[
\begin{align*}
\begin{cases}
\text{Seek } u_h \in X_h \text{ and } p_h \in M_h \text{ such that} \\
a(u_h, v_h) + b(v_h, p_h) = f(v_h), & \forall v_h \in X_h, \\
b(u_h, q_h) = g(q_h), & \forall q_h \in M_h.
\end{cases}
\end{align*}
\]

(4.13)
Proposition 4.13. The discrete problem (4.13) is well-posed if and only if the spaces $X_h$ and $M_h$ satisfy the compatibility condition

$$\exists \beta_h > 0, \inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{\int_{\Omega} q_h \nabla \cdot v_h}{\|q_h\|_{0, \Omega} \|v_h\|_{1, \Omega}} \geq \beta_h. \quad (4.14)$$

Proof. Since the bilinear form $a$ is coercive on $[H^1_0(\Omega)]^d \times [H^1_0(\Omega)]^d$ and $X_h \subset [H^1_0(\Omega)]^d$, the statement is a direct consequence of Proposition 2.42.

Using the notation of Proposition 2.42, set $B = -\nabla \cdot [H^1_0(\Omega)]^d \to L^2_{f_{=0}}(\Omega)$ and $B_h = \pi_h \nabla \cdot (i_h(\cdot)) : X_h \to M_h'$, where $i_h$ is the natural injection of $X_h$ into $[H^1_0(\Omega)]^d$ and $\pi_h$ is the $L^2$-projection from $L^2(\Omega)$ to $M_h$. Clearly,

$$\text{Ker}(B_h) = \{v_h \in X_h; \forall q_h \in M_h, b(v_h, q_h) = 0\}.$$ 

Proposition 4.14. Under hypothesis (4.14), the following estimates hold:

$$\|u - u_h\|_{1, \Omega} \leq c_{1h} \inf_{v_h \in X_h} \|u - v_h\|_{1, \Omega} + c_{2h} \inf_{q_h \in M_h} \|p - q_h\|_{0, \Omega},$$

$$\|p - p_h\|_{0, \Omega} \leq c_{3h} \inf_{v_h \in X_h} \|u - v_h\|_{1, \Omega} + c_{4h} \inf_{q_h \in M_h} \|p - q_h\|_{0, \Omega},$$

where $c_{1h} = (1 + \|a\|_\alpha)(1 + \|b\|_\alpha)$, $c_{2h} = 0$ if $\text{Ker}(B_h) \subset \text{Ker}(B)$ and $c_{2h} = \|b\|_\alpha$ otherwise, $c_{3h} = c_{1h} \frac{\|a\|}{\beta_h}$, and $c_{4h} = 1 + \|b\|_\beta + c_{2h} \frac{\|a\|}{\beta_h}$. Here, $\alpha$ is the coercivity constant of the bilinear form $\int_{\Omega} \nabla u : \nabla v$ on $[H^1_0(\Omega)]^d \times [H^1_0(\Omega)]^d$, $\|a\|$ its norm, and $\|b\|$ the norm of the bilinear form $b$ on $[H^1_0(\Omega)]^d \times L^2_{f_{=0}}(\Omega)$.

Proof. Direct application of Lemma 2.44.

Remark 4.15.

(i) In the literature, the condition (4.14) is often referred to as the Babuška–Brezzi condition [Bab73, Bre74]. The quantity $\beta_h$ is often called the inf-sup constant; see also [Pir83, GiR86, Gun89, BrF91].

(ii) The error estimates are optimal provided $\beta_h$ is bounded uniformly from below as $h$ goes to zero. Whenever possible, it is recommended to choose $X_h$ and $M_h$ so that the inf-sup constant does not depend on $h$.

(iii) In the estimate on $u - u_h$, the constants depend on $\frac{1}{\beta_h}$, whereas those in the estimate on $p - p_h$ depend on $\frac{1}{\beta_h^2}$. This means that if $\beta_h \to 0$ when $h \to 0$, the suboptimal behavior of $\beta_h$ is more damaging for the convergence rate on the pressure than that on the velocity.

(iv) It is straightforward to extend Propositions (4.13) and (4.14) to non-conformal approximation settings; see §4.2.8 for an example.

Definition 4.16 (Smoothing property). The Stokes problem (4.3) is said to have smoothing properties in $\Omega$ if assumption (ANM1) in §2.4.2 is satisfied with $H = [L^2(\Omega)]^d$, $Y = [H^2(\Omega)]^d \cap [H^1_0(\Omega)]^d$, and $N = H^1(\Omega) \cap L^2_{f_{=0}}(\Omega)$. 

Lemma 4.17. The Stokes problem has smoothing properties if one of the following statements holds:

(i) $\Omega$ is a convex polygon in two dimensions.
(ii) In two or three dimensions, $\Omega$ is of class $C^{1,1}$.

Proof. The proof is quite technical; see [Cat61, AmG91].

Proposition 4.18. Assume that:

(i) The inf-sup condition (4.14) holds.
(ii) The Stokes problem has smoothing properties.
(iii) There is $c_i$, independent of $h$, such that, for all pairs $(v, q) \in ([H_0^1(\Omega)]^d \cap [H^2(\Omega)]^d) \times (L^2_{J=0}(\Omega) \cap H^1(\Omega))$,

$$\inf_{(v_h, q_h) \in X_h \times M_h} \| v - v_h \|_{1, \Omega} + \| q - q_h \|_{0, \Omega} \leq c_i h (\| v \|_{2, \Omega} + \| q \|_{1, \Omega}).$$

Then, there is $c$ such that

$$\forall h, \quad \| u - u_h \|_{0, \Omega} \leq c h \left( \inf_{v_h \in X_h} \| u - v_h \|_{1, \Omega} + \inf_{q_h \in M_h} \| q - q_h \|_{0, \Omega} \right).$$

Proof. Apply Proposition 2.45.

4.2.2 The Fortin criterion

A powerful tool to prove the compatibility condition (4.14) is a lemma due to Fortin [For77]. This lemma is presented in an abstract form. The reader can easily rewrite it in the Stokes framework by setting $X = [H_0^1(\Omega)]^d$, $M = L^2_{J=0}(\Omega)$, and $b(v, q) = \int_{\Omega} q \nabla \cdot v$ for $v \in X$ and $q \in M$.

Lemma 4.19 (Fortin criterion). Let $X$ and $M$ be two Banach spaces and let $b \in L(X \times M; \mathbb{R})$. Assume that there is $\beta > 0$ such that the inf-sup condition

$$\inf_{q \in M} \sup_{v \in X} \frac{b(v, q)}{\| v \|_X \| q \|_M} \geq \beta$$

holds. Let $X_h \subset X$ and $M_h \subset M$, $M_h$ being reflexive. Then, there is $\beta_h > 0$ such that

$$\inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{b(v_h, q_h)}{\| v_h \|_X \| q_h \|_M} \geq \beta_h,$$  

iff there is $\gamma_h > 0$ such that, for all $v \in X$, there is $\Pi_h(v) \in X_h$ such that

$$\forall q_h \in M_h, \quad b(v, q_h) = b(\Pi_h(v), q_h) \quad \text{and} \quad \| \Pi_h(v) \|_X \leq \gamma_h \| v \|_X.$$  

Proof. (1) Assume that (4.16) holds. Let $q_h \in M_h$. Clearly,

$$\sup_{v_h \in X_h} \frac{b(v_h, q_h)}{\| v_h \|_X} \geq \sup_{v \in X} \frac{b(\Pi_h(v), q_h)}{\| \Pi_h(v) \|_X} = \sup_{v \in X} \frac{b(v, q_h)}{\| \Pi_h(v) \|_X} \geq \sup_{v \in X} \frac{b(v, q_h)}{\gamma_h \| v \|_X}.$$
The last term is bounded from below by \( \frac{\alpha}{\gamma_h} \| q_h \|_M \), proving (4.15).

(2) Let us now prove the converse. Let \( v \) be in \( X \). It is clear that \( b(v, \cdot) \in M'_h \). Define \( B_h : X_h \to M'_h \) to be the operator such that \( \langle B_h v_h, q_h \rangle_{M'_h, M_h} = b(v_h, q_h) \) for all \( v_h \in X_h \) and \( q_h \in M_h \). Since \( M_h \) is reflexive, owing to the converse statement in Lemma A.42, the inf-sup inequality implies that \( B_h \) is a surjective mapping and there is \( \Pi_h(v) \in X_h \) such that \( B_h \Pi_h(v) = b(v, \cdot) \) and \( \beta_h \| \Pi_h(v) \|_X \leq \| b(v, \cdot) \|_{M'_h} \). That is to say, for all \( v \in X \), there is \( \Pi_h(v) \) such that \( b(\Pi_h(v), q_h) = b(v, q_h) \) for all \( q_h \in M_h \) and \( \beta_h \| \Pi_h(v) \|_X \leq \| b \| \| v \|_X \). \( \square \)

### 4.2.3 Counter-examples

In this section, we study three pairs of finite element spaces that do not satisfy the inf-sup condition (4.14). This condition is not satisfied if and only if the operator \( B^T_h : M_h \to X'_h \) is not injective (or, once global shape functions have been chosen, the associated matrix has not full column rank). Equivalently, the inf-sup condition is not satisfied if and only if the operator \( B_h : X_h \to M'_h \) is not surjective. If \( B^T_h \) is not injective, a nonzero pressure field in \( \text{Ker}(B^T_h) \) is called a spurious mode.

**The \( Q_1/P_0 \) finite element: The checkerboard instability.** The most well-known pair of incompatible finite element spaces is that where the velocity is approximated by means of continuous \( Q_1 \) polynomials and the pressure by means of \( P_0 \) polynomials, i.e., piecewise constants. This pair of spaces produces the so-called checkerboard instability.

Let us restrict ourselves to a two-dimensional setting and assume that the domain is the unit square \( \Omega = [0, 1]^2 \). Define a uniform Cartesian mesh on \( \Omega \) as follows: Let \( N \) be an integer larger than 2. Set \( h = \frac{1}{N} \), and for \( 0 \leq i, j \leq N \), denote by \( a_{ij} \) the point whose coordinates are \((ih, jh)\). Let \( K_{ij} \) be the square cell whose bottom left node is \( a_{ij} \); see Figure 4.1. The resulting mesh is denoted by \( T_h = \bigcup_{i,j} K_{ij} \). Define the approximation spaces

\[
X_h = \{ u_h \in [C^0(\overline{\Omega})]^2; \forall K_{ij} \in \mathcal{T}_h, u_h \circ T_{K_{ij}} \in [Q_1]^2; u_h|_{\partial \Omega} = 0 \},
\]
\[
M_h = \{ p_h \in L^2_{\text{loc}}(\Omega); \forall K_{ij} \in \mathcal{T}_h, p_h \circ T_{K_{ij}} \in P_0 \}.
\]

Recall that for a mesh cell \( K \), \( T_K : \hat{K} \to K \) denotes the \( C^1 \)-diffeomorphism that maps the reference cell \( \hat{K} \) to \( K \); see §1.3.2. For all \( p_h \in M_h \), set \( p_{i+\frac{1}{2},j+\frac{1}{2}} = p_h|_{K_{ij}} \), and for all \( u_h \in X_h \), denote by \((u_{ij}, v_{ij})\) the values of the two Cartesian components of \( u_h \) at the node \( a_{ij} \).

To prove that the inf-sup constant is zero, it is sufficient to prove the existence of a nonzero pressure field \( p_h \in \text{Ker}(B^T_h) = \text{Im}(B_h)^{-1} \), i.e., such that \( \int_{\Omega} p_h \nabla \cdot u_h = 0 \) for all \( u_h \in X_h \). By definition of \( M_h \), \( p_h \) is constant on each cell; as a result,
Fig. 4.1. The $Q_1/P_0$ finite element and the checkerboard instability: mesh (left) and spurious mode (right).

\[
\int_{K_{ij}} p_h \nabla \cdot u_h = p_{i+\frac{1}{2}, j+\frac{1}{2}} \int_{\partial K_{ij}} u_h \cdot n
\]
\[
= \frac{1}{2} h p_{i+\frac{1}{2}, j+\frac{1}{2}} (u_{i+1,j} + u_{i+1,j+1} + v_{i+1,j+1} + v_{i,j+1} - u_{i,j} + u_{i,j+1} - v_{i,j} + v_{i+1,j}).
\]

Summing over all the cells and rearranging the sum yields

\[
\int_{\Omega} p_h \nabla \cdot u_h = -h^2 \sum_{0 \leq i,j < N} (u_{i,j} (\partial_1 p)_{ij} + v_{i,j} (\partial_2 p)_{ij}),
\]

where

\[
(\partial_1 p)_{ij} = \frac{1}{2h} (p_{i+\frac{1}{2}, j+\frac{1}{2}} + p_{i+\frac{1}{2}, j-\frac{1}{2}} - p_{i-\frac{1}{2}, j+\frac{1}{2}} - p_{i-\frac{1}{2}, j-\frac{1}{2}}),
\]
\[
(\partial_2 p)_{ij} = \frac{1}{2h} (p_{i+\frac{1}{2}, j+\frac{1}{2}} + p_{i-\frac{1}{2}, j+\frac{1}{2}} - p_{i+\frac{1}{2}, j-\frac{1}{2}} - p_{i-\frac{1}{2}, j-\frac{1}{2}}).
\]

Hence, $\int_{\Omega} p_h \nabla \cdot u_h = 0$ for all $u_h \in X_h$ if and only if for all $1 \leq i,j \leq N-1$,

\[
p_{i+\frac{1}{2}, j+\frac{1}{2}} = p_{i-\frac{1}{2}, j-\frac{1}{2}} \quad \text{and} \quad p_{i-\frac{1}{2}, j+\frac{1}{2}} = p_{i+\frac{1}{2}, j-\frac{1}{2}}.
\]

The solution set of this linear system is a two-dimensional vector space. One dimension is spanned by the constant field $p_h = 1$. But, since the elements in $M_h$ must be of zero mean, the line spanned by constant pressure fields must be excluded from the solution set. The other dimension is spanned by the field whose value is alternatively $+1$ and $-1$ on adjacent cells in a way similar to that of a checkerboard; see Figure 4.1. This oscillating function is usually referred to as a spurious mode. It is now clear that the inf-sup condition is not satisfied. Hence, the spaces $X_h$ and $M_h$ are incompatible to solve the Stokes problem.

Since the $Q_1/P_0$ finite element is very simple to program, one may be tempted to cure its deficiencies by restricting the size of $M_h$. For instance, one may enforce the pressure to be orthogonal (in the $L^2$-sense) to the space spanned by the spurious mode. Unfortunately, the cure is not strong enough to produce a healthy finite element. More precisely, it can be shown that by
Fig. 4.2. The $P_1/P_1$ finite element: the mesh (left); one spurious mode (right).

doing so, one obtains a pair of finite element spaces for which the inf-sup constant $\beta_h$ is such that

$$ch \leq \beta_h \leq c'h,$$

where the constants $c$ and $c'$ are positive and independent of $h$; see [BoN85] or [GiR86, p. 164] for further insight. This estimate shows that the method may not converge at all since $\frac{1}{\beta_h}$ comes into play in the error bound on the velocity and $\frac{1}{\beta_h^2}$ appears in the error bound on the pressure; see Proposition 4.14.

The $P_1/P_1$ finite element: Checkerboard-like instability. Because it is one of the simplest to program, the continuous $P_1$ finite element for both velocity and pressure is a natural choice for approximating the Stokes problem. Unfortunately, the $P_1/P_1$ finite element does not satisfy the inf-sup condition (4.14).

To understand the origin of the problem, let us construct a two-dimensional counterexample in the square $\Omega = [0,1]^2$. Let us consider on $\Omega$ a uniform Cartesian mesh composed of squares of side $h$. The squares are split into triangles by cutting them along one diagonal as shown in the left panel of Figure 4.2. Denote by $\mathcal{T}_h$ the resulting triangulation and define the velocity and pressure approximation spaces to be

$$X_h = \{u_h \in [C^0(\Omega)]^2; \forall K \in \mathcal{T}_h, u_h \circ T_K \in [P_1]^2; u_h|_{\partial \Omega} = 0\}; \quad (4.19)$$

$$M_h = \{p_h \in L^2(\Omega) \cap C^0(\Omega); \forall K \in \mathcal{T}_h, p_h \circ T_K \in P_1\}. \quad (4.20)$$

Given a triangle $K$, denote by $\{a_{0,K}, a_{1,K}, a_{2,K}\}$ its three vertices. Now, consider a pressure field $p_h$ such that the sum $\sum_{n=0}^2 p_h(a_{n,K})$ is zero on each triangle $K$. An example of such a spurious mode is shown in the right panel of Figure 4.2. Then,

$$\forall v_h \in X_h, \quad \int_{\Omega} p_h \nabla \cdot v_h = \sum_{K \in \mathcal{T}_h} (\nabla \cdot v_h)|_K \int_K p_h,$$

$$= \sum_{K \in \mathcal{T}_h} (\nabla \cdot v_h)|_K \frac{\text{meas}(K)}{3} \sum_{n=0}^2 p_h(a_{n,K}) = 0.$$
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The pressure field $p_h$ is such that $\int_\Omega p_h \nabla \cdot v_h$ is zero for all $v_h \in X_h$. In other words, this field is a spurious mode and the inf-sup constant is zero.

**The $P_1/P_0$ finite element: The locking effect.** A simple alternative to the $Q_1/P_0$ element consists of using the $P_1/P_0$ element. This element is appealing since it is very simple to program. Assuming that $\Omega$ is meshed with simplices, the velocity is approximated using continuous piecewise linear polynomials, and the pressure is approximated by means of (discontinuous) piecewise constants. Since the velocity is piecewise linear, its divergence is constant on each simplex. As a result, testing the divergence of the velocity by piecewise constants enforces the divergence to be zero everywhere. That is to say, the $P_1/P_0$ finite element yields a velocity approximation which is exactly divergence-free. Unfortunately, this finite element does not satisfy the inf-sup condition (4.14).

Let us produce a counterexample in two dimensions. Assume that $\Omega$ is a simply connected polygon in $\mathbb{R}^2$ and that $\Omega$ is meshed with triangles. Let $N_{el}$, $N_v$, and $N^\partial_{ed}$ be the number of elements, internal vertices, and boundary edges in the triangulation, respectively. The Euler relations yield (see Lemma 1.57)

$$N_{el} = 2N_v + N^\partial_{ed} - 2.$$  

It is clear that $\dim(M_h) = N_{el} - 1$ and $\dim(X_h) = 2N_v^i$. Let $B_h : X_h \to M_h$ be the operator such that $(B_h v_h, q_h)_{0, \Omega} = (\nabla \cdot v_h, q_h)_{0, \Omega}$ for all $v_h \in X_h$ and $q_h \in M_h$. The Rank Theorem implies

$$\dim(Ker(B^T_h)) = \dim(M_h) - \dim(\text{Im}(B^T_h)) \geq \dim(M_h) - \dim(X_h) = N_{el} - 1 - 2N_v^i = N^\partial_{ed} - 3.$$  

As a result, there are at least $N^\partial_{ed} - 3$ spurious modes. This means that the space $M_h$ is far too rich for $B_h$ to be surjective. Actually, in some cases it can be shown that $B_h$ is injective, meaning that the only solution to $B_h u_h = 0$ is $u_h = 0$. In the literature, this situation is referred to as the locking phenomenon.

**4.2.4 The $P_1$-bubble/$P_1$ finite element**

The reason for which the $P_1/P_1$ element does not satisfy the inf-sup condition (4.14) is that the velocity space is not rich enough (or, conversely, the pressure space is too rich). To circumvent this difficulty, we enlarge the velocity space. The simplest idea consists of adding one degree of freedom per element associated with the barycenter of each simplex.

Assume that $\Omega$ is a polyhedron in $\mathbb{R}^d$ and consider a sequence of affine simplicial meshes $\{T_h\}_{h>0}$. On the reference simplex $\tilde{K}$, define a function $\tilde{b}$ such that

$$\tilde{b} \in H^1_0(\tilde{K}), \quad 0 \leq \tilde{b} \leq 1, \quad \tilde{b}(\tilde{C}) = 1, \quad (4.21)$$  

where $\tilde{C}$ is the barycenter of $\tilde{K}$. Then, let
and define $\hat{\Sigma}$ to be the set of the linear forms on $\hat{P}$ that map a vector-valued function $\hat{v} \in \hat{P}$ to the value of one of its Cartesian components at one vertex of $\hat{K}$ or at the barycenter. The approximation spaces are defined as

$$X_h = \{ u_h \in [C^0(\Omega)]^d; \forall K \in T_h, u_h \circ T_K \in \hat{P}; u_h|_{\partial \Omega} = 0 \},$$ (4.22)

$$M_h = \{ p_h \in L^2_{j=0}(\Omega) \cap C^0(\Omega); \forall K \in T_h, p_h \circ T_K \in \mathbb{P}_1 \}. \quad (4.23)$$

A first possible definition of the function $\hat{b}$ consists of setting

$$\hat{b} = (d + 1)^{d+1} \prod_{i=1}^{d+1} \hat{\lambda}_i,$$

where $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_{d+1}\}$ are the barycentric coordinates on $\hat{K}$. This function is usually referred to as a bubble function in reference to the shape of its graph; see Figure 4.3. A second possibility consists of dividing the simplex $\hat{K}$ into $d + 1$ subsimplices by connecting the $d + 1$ vertices of $\hat{K}$ to its barycenter. Then, define $\hat{b}$ as the continuous, piecewise linear function equal to one at $C$ and zero at the vertices of $\hat{K}$.

**Lemma 4.20.** Let $1 < p < \infty$ and let $p'$ the conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Let $X_h$ and $M_h$ be defined in (4.22) and (4.23), respectively. If the mesh family $\{T_h\}_{h > 0}$ is shape-regular, there is $\beta$, independent of $h$, such that

$$\inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{\int_\Omega q_h \nabla \cdot v_h}{\|v_h\|_{[W^{1,p'}(\Omega)]^d} \|q_h\|_{L^{p'}(\Omega)}} \geq \beta > 0. \quad (4.24)$$

**Proof.** We apply Lemma 4.19 using Lemma B.69. Let $v$ be a function in $[W_0^{1,p'}(\Omega)]^d$. The idea is to construct $\Pi_h(v) \in X_h$ such that

$$\forall q_h \in M_h, \quad \int_\Omega q_h \nabla \cdot (\Pi_h(v)) = \int_\Omega q_h \nabla \cdot v.$$

$M_h$ being clearly a subspace of $W^{1,p'}(\Omega)$, this amounts to proving

$$\forall q_h \in M_h, \quad \sum_{K \in T_h} \int_K \Pi_h(v) \cdot \nabla q_h = \sum_{K \in T_h} \int_K v \cdot \nabla q_h.$$

Since $\nabla q_h \in [\mathbb{P}_0(K)]^d$, $\Pi_h(v)$ must be such that $\int_K \Pi_h(v) = \int_K v$ for all $K$ in $T_h$.

Let us first define an interpolant of $v$. Since $v \in [W_0^{1,p'}(\Omega)]^d$ may not be continuous, its Lagrange interpolant may not exist. However, the Clément interpolant modified to preserve homogeneous boundary conditions, $C_h(v)$, is well-defined; see Lemma 1.127 and Remark 1.129(i). Hence, it is legitimate to set

$$\tilde{P} = [\mathbb{P}_1(\hat{K}) \oplus \text{span}(\hat{b})]^d,$$
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<table>
<thead>
<tr>
<th>Velocity</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{P}_1 + \text{bubble}$</td>
<td>$3\mathbb{P}_1$ or $4\mathbb{P}_1$</td>
</tr>
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![Diagram](image)

**Fig. 4.3.** Conventional representation of the $\mathbb{P}_1$-bubble/$\mathbb{P}_1$ finite element in two dimensions (top) and in three dimensions (bottom). The degrees of freedom for the velocity are shown in the first column or in the second column ($3\mathbb{P}_1$ in two dimensions and $4\mathbb{P}_1$ in three dimensions). Some isolines of the two-dimensional bubble function are drawn. The pressure degrees of freedom are shown in the third column.

$$
\Pi_h(v) = C_h(v) + \sum_{K \in T_h} \sum_{i=1}^d \gamma^i_K e_i b_K,
$$

where $\{e_1, \ldots, e_d\}$ is the canonical basis of $\mathbb{R}^d$ and $b_K = \hat{b} \circ T_K^{-1}$. To enforce $\iiint_K \Pi_h(v) = \iiint_K v$, set

$$
\gamma^i_K = \frac{\iiint_K (v^i - C_h(v)^i)}{\iint_K b_K},
$$

where $v^i$ and $C_h(v)^i$ denote the Cartesian components of $v$ and $C_h(v)$, respectively. Since the mesh is affine, $|\det(J_K)| = \frac{\text{meas}(K)}{\text{meas}(\hat{K})}$ and

$$
\iint_K b_K = \frac{\text{meas}(K)}{\text{meas}(\hat{K})} \iint_{\hat{K}} \hat{b} = \hat{c} \text{meas}(K).
$$

Furthermore, since the family $\{T_h\}_{h>0}$ is shape-regular,

$$
\|b_K\|_{1,p,K} \leq c \|J_K^{-1}\|_d |\det(J_K)|^{\frac{1}{p}} \|\hat{b}\|_{1,p,\hat{K}} \leq c' \frac{\text{meas}(K)^{\frac{1}{2}}}{\text{meas}(\hat{K})^{\frac{1}{2}}} H^{-1}.
$$
Moreover, letting $\Delta_K$ be defined as in Figure 1.25,

$$|\gamma_K| \leq \frac{\text{meas}(K)^{\frac{1}{p'}} \|v - C_h(v)\|_{L^p(K)}}{\text{c meas}(K)} \leq c \text{meas}(K)^{\frac{1}{p}} h_K \|v\|_{W^{1,p}(\Delta_K)}.$$  

In conclusion,

$$\left\| \sum_{K \in T_h} \sum_{i=1}^{d} \gamma_i^K e_i K \right\|^p_{1,p,\Omega} \leq d_p \sum_{K \in T_h} \|b_K\|^p_{1,p,\Omega} \max\{|\gamma_1|^p, \ldots, |\gamma_d|^p\} \leq c \sum_{K \in T_h} h_K^{-p+p} \text{meas}(K)^{1+\frac{p}{p'}-p} \|v\|^p_{1,p,\Delta_K} \leq c \sum_{K \in T_h} \text{card}\{K' \in \Delta\} \|v\|^p_{1,p,\Omega}.$$  

Since the family $\{T_h\}_{h>0}$ is shape-regular, $\text{card}\{K' \in \Delta\}$ is bounded uniformly with respect to $h$. Hence, $\|\Pi_h(v)\|_{1,p,\Omega} \leq c \|v\|_{1,p,\Omega}$. Conclude using Lemma 4.19 together with Lemma B.69.

**Theorem 4.21.** Assume that the solution to the Stokes problem (4.2) is smooth enough, that is, $u \in [H^2(\Omega) \cap H^4_0(\Omega)]^d$ and $p \in H^1(\Omega) \cap L^2_{\text{div}}(\Omega)$. Then, the solution $(u_h, p_h)$ to (4.13) with $X_h$ and $M_h$ defined in (4.22) and (4.23) satisfies

$$\forall h, \quad \|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq c h (\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).$$

Moreover, if the Stokes problem has smoothing properties, then

$$\forall h, \quad \|u - u_h\|_{0,\Omega} \leq c h^2 (\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).$$

**Remark 4.22.** The idea of using bubble functions has been introduced by Crouzeix and Raviart [CrR73]. The analysis of the $\mathbb{P}_1$-bubble/$\mathbb{P}_1$ finite element is due to Arnold, Brezzi, and Fortin [ArB84]. In the literature, this element is sometimes called the mini-element.

### 4.2.5 The Taylor–Hood finite element and its generalizations

**The Taylor–Hood element: $\mathbb{P}_2/\mathbb{P}_1$.** Let us now consider a more accurate velocity approximation. We still assume that the domain $\Omega$ is a polyhedron and that $\{T_h\}_{h>0}$ is a shape-regular family of affine meshes composed of simplices. We keep the continuous $\mathbb{P}_1$ approximation for the pressure, but we approximate the velocity by means of continuous $\mathbb{P}_2$ polynomials. Accordingly, we define

$$X_h = \{u_h \in [C^0(\Omega)]^d; \forall K \in T_h, u_h \circ T_K \in [\mathbb{P}_2]^d; u_{h|\partial\Omega} = 0\}, \quad (4.25)$$

$$M_h = \{p_h \in L^2_{\text{div}}(\Omega) \cap C^0(\Omega); \forall K \in T_h, p_h \circ T_K \in \mathbb{P}_1\}. \quad (4.26)$$

The conventional representation of this element is shown in Figure 4.4.
Fig. 4.4. Conventional representation of the $P_2/P_1$ element (left) and the $Q_2/Q_1$ element (right) in two dimensions (top) and in three dimensions (bottom). In three dimensions, only visible degrees of freedom are shown.

Lemma 4.23. Assume that the space dimension $d$ is either two or three. Assume that every mesh element has at least $d$ edges in $\Omega$. Then, there is $c$ such that, for $X_h$ and $M_h$ defined in (4.25) and (4.26), the following inequality holds for all $1 < p < +\infty$:

$$
\sup_{v_h \in X_h} \frac{\int_\Omega q_h \nabla \cdot v_h}{\|v_h\|_{1,p,\Omega}} \geq c \left( \sum_{K \in T_h} h_K^{p'} [q_h]_{1,p',K} \right)^{\frac{1}{p'}}. \tag{4.27}
$$

Proof. We give the proof in three dimensions, the proof in two dimensions being similar. Number all the internal edges of the mesh from 1 to $N_{ed}^i$. For edge $i$, with $1 \leq i \leq N_{ed}^i$, denote by $d_i$ and $f_i$ its two extremities and by $m_i$ its midpoint. Set $l_i = \|f_i - d_i\|_3$ and $r_i = \frac{f_i - d_i}{\|f_i - d_i\|_3}$; $l_i$ is the length of the edge and $r_i$ is a unit vector spanning the line passing through $d_i$ and $f_i$. Let $q_h$ be a function in $M_h$ and let $\text{sgn}$ be the sign function. Define a function $v_h \in X_h$ such that, for all $K \in T_h$,

$$
\begin{align*}
\begin{cases}
  v_h = 0 & \text{at the vertices of } K, \\
  v_h(m_i) = -l_i^{p'} r_i \text{sgn}(\partial_r q_h) |\partial_r q_h|^{p' - 1} & \text{for the edges } i \text{ of } K.
\end{cases}
\end{align*}
$$

Note that the definition of $v_h(m_i)$ is consistent, i.e., this value does not depend on $K$ but only on the edge $i$. Using the quadrature formula
forall \( \phi \in \mathbb{P}_2 \), 
\[
\int_K \phi(x) \, dx = \left( \sum_m \frac{\phi(m)}{5} - \sum_n \frac{\phi(n)}{20} \right) \text{meas}(K),
\]
where \( m \) spans the set of edge midpoints of \( K \) and \( n \) spans the set of nodes of \( K \), we infer

\[
\int_\Omega q_h \nabla \cdot v_h = - \int_\Omega v_h \cdot \nabla q_h = - \sum_{K \in T_h} \int_K v_h \cdot \nabla q_h
\]

\[
= - \sum_{K \in T_h} \sum_{m_i \in K} v_h(m_i) \cdot \nabla q_h(m_i) \frac{\text{meas}(K)}{5}
\]

\[
= \sum_{K \in T_h} \sum_{m_i \in K} |\partial_t q_h(m_i)|_1^{p'} \frac{\text{meas}(K)}{5} \geq c \sum_{K \in T_h} h_K^{p'} |q_h|_{1,p',K}.
\]

The last inequality results from the fact that every tetrahedron in \( T_h \) has at least three edges in \( \Omega \), so that the quantities \( \partial_t q_h(m_i) \) where \( m_i \) spans the edge midpoints indeed control \( \nabla q_h \) on \( K \). Finally, inequality (4.27) results from

\[
\|v_h\|_{1,p,K} \leq c h_K^{p'} |q_h|_{1,p',K}. \quad \Box
\]

**Lemma 4.24.** Under the hypotheses of Lemma 4.23, the spaces \( X_h \) and \( M_h \) satisfy the inf-sup condition (4.24) uniformly with respect to \( h \).

**Proof.** Let \( q_h \) be a non-zero function in \( M_h \). According to inequality (4.5) and the converse statement in Lemma A.42, there is \( v \in [W^{1,p}_{0}(\Omega)]^d \) such that

\[
\nabla \cdot v = q_h \quad \text{and} \quad \beta \|v\|_{1,p,\Omega} \leq \|q_h\|_{0,p',\Omega}.
\]

As a result,

\[
\sup_{v_h \in X_h} \frac{\int_\Omega q_h \nabla \cdot v_h}{\|v_h\|_{1,p,\Omega}} \geq \frac{\int_\Omega q_h \nabla \cdot C_h(v)}{\|C_h(v)\|_{1,p,\Omega}} \geq c \frac{\int_\Omega q_h \nabla \cdot C_h(v)}{\|v\|_{1,p,\Omega}}
\]

\[
= c \frac{\int_\Omega q_h \nabla \cdot v}{\|v\|_{1,p,\Omega}} + c \frac{\int_\Omega q_h \nabla \cdot (C_h(v) - v)}{\|v\|_{1,p,\Omega}},
\]

where \( C_h \) is the Clément interpolation operator modified to preserve homogeneous boundary conditions; see Lemma 1.127 and Remark 1.129(i). This implies

\[
\sup_{v_h \in X_h} \frac{\int_\Omega q_h \nabla \cdot v_h}{\|v_h\|_{1,p,\Omega}} \geq c \beta \|q_h\|_{0,p',\Omega} + c \frac{\int_\Omega (C_h(v) - v) \cdot \nabla q_h}{\|v\|_{1,p,\Omega}}
\]

\[
\geq c \beta \|q_h\|_{0,p',\Omega} - c' \frac{\sum_{K \in T_h} |q_h|_{1,p',K} h_K \|v\|_{1,p,\Delta_K}}{\|v\|_{1,p,\Omega}}
\]

\[
\geq c \beta \|q_h\|_{0,p',\Omega} - c' \left( \sum_{K \in T_h} h_K^{p'} |q_h|_{1,p',K} \right)^\frac{1}{p'}.
\]

Owing to Lemma 4.23, the negative term is bounded from below as follows:
4.2. Mixed Finite Element Approximation

This yields the expected inequality with the constant $\frac{c_1}{1+c_2}$.

**Remark 4.25.** For other details and alternative proofs, the reader is referred to [BeP79, pp. 255–257] and [GiR86, p. 176]. The main ideas of the above proof are adapted from [Ver84], but the extension to $L^p$ is original. In the literature, the $P_2/P_1$ finite element is also known as the Taylor–Hood element.

**Theorem 4.26.** Assume that the solution $(u, p)$ to the Stokes problem (4.2) is smooth enough, that is, $u \in [H^3(\Omega) \cap H^1_0(\Omega)]^d$ and $p \in H^2(\Omega) \cap L^2_0(\Omega)$. Then, the solution $(u_h, p_h)$ to (4.13) with $X_h$ and $M_h$ defined in (4.25) and (4.26) satisfies

$$\forall h, \|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq c h^2 (\|u\|_{3,\Omega} + \|p\|_{2,\Omega}).$$

Moreover, if the Stokes problem has smoothing properties, then

$$\forall h, \|u - u_h\|_{0,\Omega} \leq c h^3 (\|u\|_{3,\Omega} + \|p\|_{2,\Omega}).$$

**The $P_k/P_{k-1}$ and $Q_k/Q_{k-1}$ finite elements.** Still keeping a continuous approximation of the pressure, it is possible to generalize the Taylor–Hood element to quadrilaterals and hexahedra. For instance, the $Q_2/Q_1$ finite element has the same properties as those of the Taylor–Hood element; see Figure 4.4.

It is also possible to use higher-degree polynomials. For $k \geq 2$, the $P_k/P_{k-1}$ finite elements ($P_k$ for velocity and $P_{k-1}$ for pressure) as well as the $Q_k/Q_{k-1}$ finite elements ($Q_k$ for velocity and $Q_{k-1}$ for pressure) are compatible in two and three dimensions. These elements yield the errors estimates

$$\|u - u_h\|_{0,\Omega} + h(\|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega}) \leq c h^{k+1} (\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega}),$$

provided the exact solution is smooth enough. Proofs and further insight can be found in [BrF91a, BrF91b].

4.2.6 The $P_1$-iso-$P_2/P_1$ finite element and its generalizations

**The $P_1$-iso-$P_2/P_1$ finite element.** An alternative to the $P_2/P_1$ finite element is to replace the $P_2$ approximation of the velocity by a $P_1$ approximation on a finer mesh. Again, assume that $\Omega$ is a polyhedron and that the family $\{T_h\}_{h>0}$ is shape-regular and composed of affine simplices. We construct a new mesh $T_{3/2}$ as follows. In dimension 2, we divide each triangle of $T_h$ into four new triangles by connecting the midpoint of the edges. In dimension 3, we divide each tetrahedron of $T_h$ into eight new tetrahedra by dividing each face into four new triangles like in two dimensions, and by connecting the midpoints of one pair of non-intersecting edges; see bottom left panel in Figure 4.5. The approximation spaces are defined as follows:
Fig. 4.5. $\mathbb{P}_1$-iso-$\mathbb{P}_2/\mathbb{P}_1$ (left) and $\mathbb{Q}_1$-iso-$\mathbb{Q}_2/\mathbb{Q}_1$ (right) finite elements in two dimensions (top) and in three dimensions (bottom). In three dimensions, only visible degrees of freedom are shown.

This finite element is often called $\mathbb{P}_1$-iso-$\mathbb{P}_2/\mathbb{P}_1$ or also $4\mathbb{P}_1/\mathbb{P}_1$ in two dimensions and $8\mathbb{P}_1/\mathbb{P}_1$ in three dimensions.

**Lemma 4.27.** The spaces $X_h$ and $M_h$ defined in (4.28) and (4.29) satisfy the inf-sup condition (4.24) uniformly with respect to $h$.

**Proof.** Easy adaptation of Lemma 4.24; see [BeP79] for the analysis in dimension 2.

**Theorem 4.28.** Assume that the solution to the Stokes problem (4.2) is smooth enough, that is, $u \in [H^2(\Omega) \cap H^1_0(\Omega)]^d$ and $p \in H^1(\Omega) \cap L^2_{\mathrm{div}}(\Omega)$. Then, the solution to (4.13) with $X_h$ and $M_h$ defined in (4.28) and (4.29) satisfies

$$\forall h, \quad \|u - u_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq c h (\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).$$

Moreover, if the Stokes problem has smoothing properties, then

$$\forall h, \quad \|u - u_h\|_{0,\Omega} \leq c h^2 (\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).$$
To approximate the velocity and the pressure, introduce the spaces

\[
X_h = \{ u_h \in [C^0(\Omega)]^d; \forall K \in T_h, u_h \circ T_K \in \hat{P}; u_h|_{\partial \Omega} = 0 \},
\]
\[
M_h = \{ p_h \in L^2_{f=0}(\Omega); \forall K \in T_h, p_h \circ T_K \in \mathbb{P}_1 \}.
\]

(4.32) \hspace{1cm} (4.33)

Note that the pressure is locally \( \mathbb{P}_1 \) on each simplex but is not necessarily continuous across the interfaces of the simplices. The local degrees of freedom for the pressure can be taken to be its mean-value and its gradient. Note also that the approximation space \( X_h \times M_h \) is conformal in \([H^1_0(\Omega)]^d \times L^2_{f=0}(\Omega)\). This finite element is often called the conformal Crouzeix–Raviart mixed finite element [CrR73]. A conventional representation is shown in Figure 4.7.

The \( P_2\)-bubble/\( P_1 \)-discontinuous element satisfies the inf-sup condition uniformly with respect to \( h \) and yields error estimates that are identical to those of the Taylor–Hood element; see [BrF91b, p. 214].

The \( Q_2/\mathbb{P}_1 \)-discontinuous finite element. It is possible to generalize the conformal Crouzeix–Raviart mixed finite element to cuboids. Let \( \Omega \) be a polyhedron in \( \mathbb{R}^d \) and let \( \{ T_h \}_{h>0} \) be a shape-regular family of meshes composed of cuboids. Introducing the spaces

\[
X_h = \{ u_h \in [C^0(\Omega)]^d; \forall K \in T_h, u_h \circ T_K \in [Q_2]^d; u_h|_{\partial \Omega} = 0 \},
\]
\[
M_h = \{ p_h \in L^2_{f=0}(\Omega); \forall K \in T_h, p_h \circ T_K \in \mathbb{P}_1 \},
\]

(4.34) \hspace{1cm} (4.35)

one obtains a mixed finite element that satisfies the inf-sup condition uniformly in \( h \), and yields the same error estimates as those of the Taylor–Hood element. As before, the local degrees of freedom for the pressure can be taken to be its mean-value over an element and that of its gradient; see [BrF91b, pp. 216–219].

Remark 4.29. Note that the \( Q_2/Q_1 \)-discontinuous finite element does not satisfy the inf-sup condition. \( \square \)
The $Q_1$-iso-$Q_2/Q_1$ finite element. It is possible to generalize the concept of the $P_1$-iso-$P_2/P_1$ finite element to quadrangles in dimension 2 and hexahedra in dimension 3.

Assume that $\Omega$ is a polygon in $\mathbb{R}^2$ (resp., polyhedron in $\mathbb{R}^3$). Let $\{T_h\}_{h>0}$ be a shape-regular family of meshes composed of quadrangles or hexahedra. Let us construct a new mesh $T_h^{2}$ as follows. In dimension 2, we divide each quadrangle of $T_h$ into four new quadrangles by connecting the midpoints of non-intersecting edges. In dimension 3, we divide each hexahedron of $T_h$ into eight new hexahedra by dividing each face into four quadrangles and by connecting the midpoint nodes of non-intersecting faces; see right panels in Figure 4.5. The velocity and pressure approximation spaces are defined to be

$$X_h = \{u_h \in [C^0(\overline{\Omega})]^d; \forall K \in T_h, u_h \circ T_K \in [Q_1]^d; u_h|_{\partial \Omega} = 0\}, \quad (4.30)$$

$$M_h = \{p_h \in L^2_{f=0}(\Omega) \cap C^0(\overline{\Omega}); \forall K \in T_h, p_h \circ T_K \in Q_1\}. \quad (4.31)$$

This finite element is often called $Q_1$-iso-$Q_2/P_1$ or $4Q_1/Q_1$ in dimension 2 (resp., $8Q_1/Q_1$ in dimension 3).

The $Q_1$-iso-$Q_2/Q_1$ finite element satisfies the inf-sup condition uniformly with respect to $h$ and yields the same error estimates as those of the $P_1$-iso-$P_2/P_1$ element; see Proposition 4.28. This type of element is often used in the industry since it is simple to implement. Figure 4.6 shows an $8Q_1/Q_1$ mesh of pipes. Only the $Q_1$ hexahedra approximating the pressure are shown.

4.2.7 Discontinuous approximation of the pressure

The $P_2$-bubble/$P_1$-discontinuous finite element. Consider a shape-regular family of affine simplicial meshes of $\Omega$, say $\{T_h\}_{h>0}$. Consider the bubble function $\hat{b}$ defined in (4.21) and set

$$\hat{P} = [P_2(\hat{K}) \oplus \text{span}(\hat{b})]^d.$$
4.2.8 The $\mathbb{P}_1$-non-conformal/$\mathbb{P}_0$ finite element

We now present a non-conformal approximation technique for the velocity based on the non-conformal Crouzeix–Raviart finite element studied in §3.2.3.

Let $\{T_h\}_{h>0}$ be a shape-regular family of affine triangulations of a domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3. Let $\mathcal{F}_h^i$ be the set of internal faces. Let $\mathcal{F}_h^0$ be the set of faces at the boundary and set $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^0$. For $F \in \mathcal{F}_h^i$ with $F = K_1 \cap K_2$, denote by $n_1$ and $n_2$ the outward normal to $K_1$ and $K_2$, respectively. Let $v$ be an $\mathbb{R}^d$-valued function that is smooth enough to have limits on both sides of $F$ (these limits being not necessarily the same). Set $v_1 = v|_{K_1}$ and $v_2 = v|_{K_2}$, and define the jump of $v$ across $F$ to be $[v] = v_1 \otimes n_1 + v_2 \otimes n_2$. Define the velocity and pressure spaces to be

$$X_h = \{v_h; \forall K \in T_h, v_h|_K \in [\mathbb{P}_1]^d; \forall F \in \mathcal{F}_h^i, \int_F [v_h] = 0; \forall F \in \mathcal{F}_h^0, \int_F v_h = 0\},$$

$$M_h = \{q_h; \forall K \in T_h, q_h|_K \in \mathbb{P}_0; \int_\Omega q_h = 0\}.$$

It is clear that functions in $X_h$ are continuous at the center of the interfaces and are zero at the center of those faces that are at the boundary. The conventional representation of this element is shown in Figure 4.8. Let

$$a_h(v_h, w_h) = \sum_{K \in T_h} \int_K \nabla v_h : \nabla w_h, \quad b_h(v_h, q_h) = -\sum_{K \in T_h} \int_K q_h \nabla \cdot v_h,$$

and, for $1 < p < +\infty$, equip $X_h$ with the mesh-dependent norm $\|v_h\|_{1,p,h,\Omega} = \sum_{K \in T_h} \|v_h\|_{1,p,K}^p$. Assuming $f \in [L^2(\Omega)]^d$, the approximate Stokes problem is:

$$\begin{align*}
\text{Seek } u_h \in X_h \text{ and } p_h \text{ in } M_h \text{ such that } \\
\begin{cases}
 a_h(u_h, v_h) + b_h(v_h, p_h) = \int_\Omega f v_h, & \forall v_h \in X_h, \\
b_h(u_h, q_h) = -\int_\Omega g q_h, & \forall q_h \in M_h.
\end{cases}
\end{align*} \tag{4.36}$$

<table>
<thead>
<tr>
<th>Dimension 2</th>
<th>Dimension 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity</td>
<td>velocity</td>
</tr>
<tr>
<td>pressure</td>
<td>pressure</td>
</tr>
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</table>

Fig. 4.8. Conventional representation of the $\mathbb{P}_1$-non-conformal/$\mathbb{P}_0$ finite element in two (left) and three (right) dimensions. In three dimensions, only visible degrees of freedom for the velocity are shown. The pressure degree of freedom is its average over the mesh cell.
To prove that the spaces pair \( \{X_h, M_h\} \) satisfies the inf-sup condition, we need to introduce some technicalities. Let \( F \) be a face and denote by \( m_F \) the center of \( F \). Then, let \( P_{pt,h}^1 \) be the Crouzeix–Raviart finite element space introduced in §1.4.3, and define \( \Pi_h : W^{1,p}(\Omega) \to P_{pt,h}^1 \) to be the \( \mathbb{P}_1 \) interpolation operator such that

\[
\forall \phi \in W^{1,p}(\Omega), \forall F \in \mathcal{F}_h, \quad \Pi_h(\phi)(m_F) = \frac{1}{\text{meas}(F)} \int_F \phi.
\]

**Lemma 4.30.** Let \( 1 < p < \infty \). There is \( c \), independent of \( h \), such that

\[
\forall \phi \in W^{1,p}(\Omega), \quad \|\Pi_h(\phi)\|_{1,p,h,\Omega} \leq c\|\phi\|_{1,p,\Omega}. \tag{4.37}
\]

**Proof.** Let \( \{F_0, \ldots, F_d\} \) be the faces of \( K \) and denote by \( \{\phi_0, \ldots, \phi_d\} \) the mean-values of \( \phi \) on \( \{F_0, \ldots, F_d\} \), respectively. Set \( \tilde{\phi} = \phi - \phi_0 \). Since constants are invariant under \( \Pi_h \),

\[
|\Pi_h\phi|_{1,p,K} = |\Pi_h\tilde{\phi}|_{1,p,K} \leq c\|J_K^{-1}\|_d|\det(J_K)|^{\frac{1}{p}}|\Pi_h\tilde{\phi}|_{1,p,\tilde{K}} \leq c\|J_K^{-1}\|_d|\det(J_K)|^{\frac{1}{p}} \max_{0 \leq j \leq d} |\tilde{\phi}_j|.
\]

Furthermore, \( |\tilde{\phi}_j| \leq c\|\tilde{\phi}\|_{0,p,\partial\tilde{K}} \leq c'\|\phi\|_{1,p,\tilde{K}} \). Now, recall the Poincare–Friedrichs inequality (see Lemma B.63)

\[
\exists c > 0, \forall \psi \in H^1(\tilde{K}), \quad c\|\psi\|_{1,p,\tilde{K}} \leq |\psi|_{1,p,\tilde{K}} + \left|\int_{F_0} \psi\right|. \tag{4.38}
\]

Using (4.38) together with the fact that \( \int_{F_0} \tilde{\phi} = 0 \) yields \( |\tilde{\phi}_j| \leq c\|\tilde{\phi}\|_{1,p,\tilde{K}} \). As a result,

\[
|\Pi_h\phi|_{1,p,K} \leq c\|J_K^{-1}\|_d|\det(J_K)|^{\frac{1}{p}}|\phi|_{1,p,\tilde{K}} \leq c'\|\phi\|_{1,p,K} = c'\|\phi\|_{1,p,K}.
\]

The rest of the proof follows easily. \( \square \)

**Lemma 4.31.** There is \( \beta \), independent of \( h \), such that

\[
\inf_{q_h \in M_h} \sup_{v_h \in X_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{1,p,h,\Omega}\|q_h\|_{0,p',\Omega}} \geq \beta. \tag{4.39}
\]

**Proof.** Let \( v \in [W_0^{1,p}(\Omega)]^d \), \( q_h \in M_h \), and \( K \in \mathcal{T}_h \). It is clear that

\[
\int_K q_h \nabla \cdot v = q_h \sum_{i=0}^d n_i \int_{F_i} \nabla \cdot v = q_h \sum_{i=0}^d n_i \int_{F_i} \Pi_h(v) = \int_K q_h \nabla \cdot \Pi_h(v).
\]

Hence, \( b_h(v, q_h) = b_h(\Pi_h v, q_h) \). Then, using Lemma 4.30 and adapting slightly the proof of Fortin’s Lemma, the conclusion follows. \( \square \)
Theorem 4.32. Assume that the solution to the Stokes problem (4.2) is smooth enough, that is, \( u \in [H^2(\Omega) \cap H^1_0(\Omega)]^d \) and \( p \in H^1(\Omega) \cap L^2_{f=0}(\Omega) \). Then, the solution to (4.36) satisfies
\[
\forall h, \quad \|u - u_h\|_{1,h,\Omega} + \|p - p_h\|_{0,\Omega} \leq c h (\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).
\]
Moreover, if the Stokes problem has smoothing properties, then
\[
\forall h, \quad \|u - u_h\|_{0,\Omega} \leq c h^2 (\|u\|_{2,\Omega} + \|p\|_{1,\Omega}). \tag{4.40}
\]

Proof. See Exercise 4.6.

Remark 4.33. (i) The \( P_1\)-non-conformal/\( P_0 \) finite element has been introduced by Crouzeix and Raviart [CrR73] and is often called the non-conformal Crouzeix–Raviart mixed finite element. A quadrilateral non-conformal mixed finite element has been introduced by Rannacher and Turek [RaT92, Tur99].

(ii) Non-conformal mixed finite elements can be used to construct piecewise divergence-free approximation spaces, i.e., non-conformal approximations to the constrained problem (4.7); see Hecht [Hec81, Hec84], Braess [Bra97, p. 154], or Brezzi and Fortin [BrF91b, p. 268] for further insight.

4.2.9 Numerical illustration

We conclude this section with a brief numerical illustration. Consider the two-dimensional O-shaped domain shown in the left panel of Figure 4.9. We perform time-dependent simulations of the Navier–Stokes equations with the \( P_1/P_1 \) and the Taylor–Hood finite elements. The Reynolds number is \( Re = 100 \) and the time step is \( \delta t = 0.01 \). The initial condition is the flow at rest. The flow is then driven anticlockwise by imposing a vertical velocity on the two outer faces. Pressure isolines after 100 time steps are shown in the central panel (resp., right panel) of Figure 4.9 for the \( P_1/P_1 \) (resp., Taylor–Hood) finite element. The pressure field obtained with the \( P_1/P_1 \) element is polluted by spurious oscillations. This example clearly shows the adverse effects of violating the inf-sup condition.