AMS507 Introduction to Probability - Midterm II
Fall 2009

Your name:
Your ID number:

You shall receive 10 points for each of the following 10 question. The total will be divided by 10. The maximum score is 10.

1. The random variable X has probability density function

$$f(x) = \begin{cases} |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E[X]$ and $Var(X)$.

$$E[X] = \int_{-1}^{1} x \cdot |x| \, dx = 0$$

$$E[X^2] = \int_{-1}^{1} x^2 |x| \, dx = \int_{-1}^{0} -x^3 dx + \int_{0}^{1} x^3 dx = \frac{1}{2}$$

$$Var(X) = E[X^2] - E^2[X] = \frac{1}{2}$$

2. Suppose that X is uniformly distributed in $[-2, 2]$. Find the probability density function of the random variable $Y = X^6$.

The range of Y is $[0, 2^6]$

$$P(Y \leq y) = P(X^6 \leq y) = P(-y^{\frac{1}{6}} \leq X \leq y^{\frac{1}{6}})$$

$$= \int_{-y^{\frac{1}{6}}}^{y^{\frac{1}{6}}} \frac{1}{4} dx = \frac{y^{\frac{1}{6}}}{2} \quad y \in [0, 2^6]$$

$$f_Y(y) = \frac{d F(y)}{dy} = \frac{1}{12} y^{-\frac{5}{6}} \quad y \in [0, 2^6]$$

3. Let X and Y be independent geometric random variables with parameters 0.5 and 0.25, respectively. Find $P(X = Y)$.

$$P(X=k) = \left(\tfrac{1}{2}\right)^{k-1} \cdot \tfrac{1}{2}, \quad P(Y=k) = \left(\tfrac{3}{4}\right)^{k-1} \cdot \tfrac{1}{4}$$

$$P(X=Y) = \sum_{k=1}^{\infty} P(X=k, Y=k)$$

$$= \sum_{k=1}^{\infty} P(X=k)P(Y=k) = \sum_{k=1}^{\infty} \left(\tfrac{3}{8}\right)^{k-1} \cdot \tfrac{1}{8}$$

$$= \frac{1}{8} \cdot \frac{1}{1-\frac{3}{8}} = \frac{1}{5}$$

4. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that, starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?

Let X = annual rainfall

$$X \sim N(40, 16)$$

$$P(X > 50) = P\left(\frac{X-40}{4} > \frac{10}{4}\right) = 1 - \phi(2.5) = 1 - 0.9938$$

N = # of years required before a year having a rainfall of over 50 inches

$$P(N=10) = (0.9938)^{10} \cdot (1-0.9938)$$

5. I am selling my house, and have decided to accept the first offer exceeding $\$K$. Assuming that offers are independent exponential random variables with common mean $\$K$. What is the expected number of offers received *until* I sell the house (including the offer that is finally accepted)?

p = The prob of accepting an offer.

X = the offer amount

$p = P(X > K) = e^{-1}$, where $X \sim \exp(\frac{1}{K})$

Let Y = # of offers received ~~before~~ until the house is sold.

$E[Y] = \frac{1}{p} = e$

6. Consider a system A consisting of two parallel components, each has an exponentially distributed lifetime with parameter λ. Assume that the lifetimes of these two components are independent, and the system works if at least one component is working. System B consists of only one component, whose lifetime is exponential distributed with parameter μ. Find the probability that system A fails before system B.

Let X_i = lifetime of the ith component in system A. $i=1,2$

$X_i \sim \exp(\lambda)$. X = lifetime of system A.

Y = lifetime of system B, $Y \sim \exp(\mu)$

$P(X \le x) = P(\max\{X_1, X_2\} \le x) = [P(X_1 < x)]^2 = (1-e^{-\lambda x})^2$

$f_X(x) = 2(1-e^{-\lambda x})\lambda e^{-\lambda x} \quad x \ge 0$

$$P(X<Y) = \iint_{x<y} 2(1-e^{-\lambda x})\lambda e^{-\lambda x} \cdot \mu e^{-\mu y}\, dx\, dy$$

$$= \int_0^\infty \left[\int_0^y 2(1-e^{-\lambda x})\lambda e^{-\lambda x}\, dx\right] \cdot \mu e^{-\mu y}\, dy$$

$$= \frac{2\lambda^2}{(\mu+\lambda)(\mu+2\lambda)}$$

7. Let X and Y have the joint density function

$$f(x,y) = \begin{cases} cx(y-x)e^{-y} & 0 \le x \le y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of c and the conditional probability density function $f_{X|Y}(x|y)$.

$$c \cdot \int_0^\infty \int_0^y x(y-x)\, e^{-y}\, dx\, dy = 1 \Rightarrow c \cdot \int_0^\infty \frac{y^3}{6} e^{-y}\, dy = 1$$

$$\Rightarrow c = 1$$

$$f_Y(y) = \int_0^y x(y-x)e^{-y}\, dx = \frac{y^3}{6} e^{-y} \qquad 0 < y < \infty$$

$$f_{X|Y}(x|y) = \frac{6x(y-x)}{y^3} \qquad 0 \le x \le y$$

8. The joint density function of X and Y is given by $f(x,y) = xe^{-x(y+1)}$ $x > 0$, $y > 0$. Find the density function of $Z = XY$.

$$P(Z \le z) = P(XY \le z)$$

$$= \int_0^\infty \int_0^{z/x} x\, e^{-x(y+1)}\, dy\, dx$$

$$= \int_0^\infty (1 - e^{-z}) e^{-x}\, dx$$

$$= 1 - e^{-z} \qquad z > 0.$$

$$f_Z(z) = e^{-z}, \quad z > 0$$

9. Let X and Y be independent exponential random variables with parameters λ and μ. Let $Z = \min\{X, Y\}$. Find $P(X = Z)$.

$$P(X=Z) = P(X = \min\{X,Y\}) = P(X \le Y)$$
$$= \iint_{x<y} f_{XY}(x,y)\,dx\,dy$$
$$= \int_0^\infty \left[\int_0^y \lambda e^{-\lambda x}\,dx\right] \mu e^{-\mu y}\,dy$$
$$= \frac{\lambda}{\lambda+\mu}$$

10. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} (\frac{2}{\pi})^{\frac{1}{2}} e^{-\frac{x^2}{2}} & \text{if } x \ge 0 \\ 0 & \text{otherwise,} \end{cases}$$

Find $E[X^2]$.

$$E[X^2] = \int_0^\infty x^2 \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

Since $x^2 e^{-\frac{x^2}{2}}$ is an even function

$$E[X^2] = 2 \cdot \int_0^\infty x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$
$$= \int_{-\infty}^\infty x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$
$$= E[Y^2] \quad \text{where } Y \sim N(0,1)$$
$$= 1.$$