AMS 553: Homework 3

1. (L&K 8.7) For $a < b$, the right-triangular distribution has density function

$$f_R(x) = \begin{cases} \frac{2(x-a)}{(b-a)^2} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

and the left-triangular distribution has density function

$$f_L(x) = \begin{cases} \frac{2(b-x)}{(b-a)^2} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

These distributions are denoted by $RT(a,b)$ and $LT(a,b)$, respectively.

   (a) Show that if $X \sim RT(0,1)$, then $X' = a + (b-a)X \sim RT(a,b)$; verify the same relation between $LT(0,1)$ and $LT(a,b)$. Thus it is sufficient to generate from $RT(0,1)$ and $LT(0,1)$.

   (b) Show that if $X \sim RT(0,1)$, then $1 - X \sim LT(0,1)$. Thus it is enough to restrict our attention further to generating from $RT(0,1)$.

   (c) Derive the inverse-transform algorithm for generating from $RT(0,1)$. Despite the result in (*b*), also derive the inverse-transform algorithm for generating directly from $LT(0,1)$.

   (d) As an alternative to the inverse-transform method, show that if U_1 and U_2 are i.i.d. $U(0,1)$ random variables, then $\max\{U_1, U_2\} \sim RT(0,1)$. Do you think that this is better than the inverse-transform method?

2. (L&K 8.8) In each of the following case, give an algorithm that uses exactly *one* random number for generating a random variate with the same distribution as X.

   (a) $X = \min\{U_1, U_2\}$, where U_1 and U_2 are i.i.d. $U(0,1)$.

   (b) $X = \max\{U_1, U_2\}$, where U_1 and U_2 are i.i.d. $U(0,1)$.

   (c) $X = \min\{Y_1, Y_2\}$, where Y_1 and Y_2 are i.i.d. exponential with common mean β.

3. (L&K 8.9) Let X be discrete with p.m.f. $p(x_i)$ for $i = 0, \pm 1, \pm 2, \ldots$. The discrete acceptance rejection method for generating X is as the following: let the majorizing function be $t_i(x_i) \ge p(x_i)$ for all i, let $c = \sum_{i=-\infty}^{\infty} t(x_i)$, and let $r(x_i) = t(x_i)/c$ for $i = 0, \pm 1, \pm 2, \ldots$.

   (a) Generate Y from p.m.f. r.

   (b) Generate $U \sim U(0,1)$, independent of Y.

   (c) If $U \le p(Y)/t(Y)$, return $X = Y$. Otherwise, go back to step 1 and try again.

   Show that this algorithm is valid. What considerations are important in choosing the function $t(x_i)$?

4. Consider a random variable X with density function

$$f(x) = \begin{cases} 0.5 & \text{if } -k < x < 0 \\ x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

   (a) Find the value of K. Find the minimum and maximum values of X. Compute $E[X]$ and the probability $P(|X - E[X]| > 1)$.

   (b) Use the inverse transform method, composition, and acceptance rejection approach to generate X. What is the efficiency of the acceptance rejection method? Discuss which of the three algorithms is preferable.

5. Let X and Y be i.i.d. $U(0,1)$. Let $Z = min(X,Y)$ and $W = X + Y$.

   (a) Find the cumulative distribution function for Z and W.

   (b) For each of these two random variable Z and W, give two different algorithms to generate the random variates. One algorithm that uses exactly one random number per random variate, the other algorithm may use more than one random numbers per random variate.