Applied Calculus I
Practice Final Exam – Solution Notes

1. Solve for x: $10 \cdot 3^{2x} = 3 \cdot (1.12)^x$

Taking the natural log of both sides, we get

$$\ln 10 + 2x \ln 3 = \ln 3 + x \ln 1.12$$

$$x(2\ln 3 - \ln 1.12) = \ln 3 - \ln 10$$

$$x = \frac{\ln 3 - \ln 10}{2\ln 3 - \ln 1.12} = \frac{\ln(3/10)}{\ln(9/1.12)}$$

2. Determine which function has a larger value as $x \to \infty$:

(a). $f(x) = 0.0005 \cdot x^3$ or $g(x) = 52,000 \cdot 2^x$ $f(x)$ grows like x^3 (we can ignore the constant factor).

$g(x)$ grows like 2^x, and exponential function (again, we can ignore the constant factor).
Thus, $g(x)$ has a larger value as $x \to \infty$.
Another way to say this is that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

so that g dominates f as $x \to \infty$, which we compute using l'Hopital's rule (since it has form $\frac{\infty}{\infty}$):

$$\lim_{x\to\infty} \frac{0.0005 \cdot x^3}{52,000 \cdot 2^x} = \lim_{x\to\infty} \frac{0.0005 \cdot 3x^2}{52,000 \cdot 2^x \ln 2} = \lim_{x\to\infty} \frac{0.0005 \cdot 6x}{52,000 \cdot 2^x \ln 2 \ln 2} = \lim_{x\to\infty} \frac{0.0005 \cdot 6}{52,000 \cdot 2^x \ln 2 \ln 2 \ln 2} = 0.$$

(b). $f(x) = \log x^7$ or $g(x) = 6 \cdot \log^2 x^2$ $f(x)$ grows like $\log x$ (since $\log x^7 = 7 \log x$), while $g(x)$ grows like $\log^2 x$ (since $\log^2 x^2 = 2\log^2 x$), so $g(x)$ dominates $f(x)$ as $x \to \infty$.

3. Find the equation of the line that goes through the point (8,3) and is perpendicular to the plot of the equation $\frac{2x+1}{y-1} = 2$.

We rewrite $\frac{2x+1}{y-1} = 2$: $2y - 2 = 2x + 1$, so that it is the equation of the line $y = x + \frac{3}{2}$. The slope of this line is 1; thus, the slope of a line perpendicular to it is $\frac{-1}{1} = -1$.

Thus, the equation of the line, L, we desire is $y = -x + b$. We find b by requiring that (8,3) lies on line L: $3 = -8 + b$, so $b = 11$. Thus, line L has equation $y = -x + 11$.

4. The quantity of moisture in a loaf of bread decreases with time as it sits on the table. Suppose that the moisture, $M(t)$, at time t minutes after being placed on the table decreases according to the function $M(t) = Qe^{-kt}$. If 10% of the moisture is gone at the end of 1 hour,

(a). What percentage of the original moisture is present after 30 minutes?

We are told that $M(60) = (.9)M(0)$, which implies that

$$Qe^{-60k} = (.9)Q$$

$$-60k = \ln(.9)$$

$$k = \frac{-\ln(.9)}{60}$$

Now, we want to find

$$\frac{M(30)}{M(0)} \cdot 100 = \frac{Qe^{-30k}}{Q} \cdot 100 = 100e^{-30\frac{-\ln(.9)}{60}} = 100e^{\frac{1}{2}\ln(.9)} = 100(.9)^{1/2}$$

(b). How long will it take until the moisture is reduced to 60% of its original quantity?

We want to find t so that $M(t) = 0.6M(0) = 0.6Q$. Thus, we want t so that $Qe^{-kt} = 0.6Q$, i.e., so that $e^{-kt} = 0.6$. Thus, we want t so that $-kt = \ln(0.6)$, i.e.,

$$t = \frac{\ln(0.6)}{-k} = \frac{\ln(0.6)}{\frac{\ln(0.9)}{60}} = 60\frac{\ln(0.6)}{\ln(0.9)}$$

5. Evaluate the following expressions:
(a). $\cos(\sin^{-1}(\frac{5}{11})) =$

Draw a right triangle and let θ be one of the two angles that is not $\pi/2$. Then, $\theta = \sin^{-1}(\frac{5}{11})$ if the side opposite to θ is of length 5 and the hypotenuse is of length 11. Then, the side adjacent to θ is of length $\sqrt{11^2 - 5^2} = \sqrt{96}$. Thus, $\cos(\sin^{-1}(\frac{5}{11})) = \cos\theta = \frac{\sqrt{96}}{11}$.

(b). $\cos(\frac{32\pi}{3}) =$ Note that $\frac{32\pi}{3} = 10\pi + \frac{2\pi}{3}$ (i.e., it is 10 revolutions, plus an additional angle $2\pi/3$ (120 degrees). Thus, $\cos(\frac{32\pi}{3}) = \cos(2\pi/3) = -\frac{1}{2}$. (Draw a picture! The angle refers to a point in the second quadrant.)

6. On February 10, 1990, high tide in Boston was at midnight. The water level at high tide was 9.9 feet; later, at low tide, it was 0.1 feet. Assuming the next high tide is at exactly 12 noon and that the height of the water is given by a sine or cosine curve, find a formula for the water level (in feet) in Boston as a function of time t, measured in hours from midnight.

Let $H(t)$ be the height in feet of the tide in Boston t hours after midnight of February 10, 1990.
Since $H(t)$ oscillates between 0.1 and 9.9, we know that the amplitude is (9.9-0.1)/2=4.9. The baseline of the oscillation occurs at $H = 0.1 + 4.9 = 5$; i.e., it is a sin/cos shifted upwards by 5.
Since the high value (max) occurs at $t = 0$ and the first high (max) after that occurs at $t = 12$ (noon), we know that the period is 12 hours.
Draw a picture! We can view $H(t)$ as a cos function that is shifted upwards by 4.9:

$$H(t) = 5 + 4.9\cos(\frac{2\pi}{12}t)$$

7. Find the asymptotes for the following function:

$$y = \frac{13 - 9x + 5x^2}{2x^2 - 1}$$

(a). Vertical: The denominator, $2x^2 - 1$, vanishes at $x = \sqrt{1/2}$ and at $x = -\sqrt{1/2}$, so these two values of x define the two vertical asymptotes.

(b). Horizontal:
We can rewrite (by dividing numerator and denominator by x^2)

$$y = \frac{\frac{13}{x^2} - \frac{9}{x} + 5}{2 - \frac{1}{x^2}}$$

from which we see the behavior as x go to infinity. As $x \to +\infty$, y goes up to 5/2. As $x \to -\infty$, y goes down to 5/2.
Thus, $y = 5/2$ is a horizontal asymptote.

(c). Sketch a plot of $y(x)$

Note that for $x = \sqrt{1/2} + \epsilon$ (for tiny $\epsilon > 0$), y is positive, heading to $+\infty$. Note that for $x = \sqrt{1/2} - \epsilon$ (for tiny $\epsilon > 0$), y is negative, heading to $-\infty$.
Note that for $x = -\sqrt{1/2} + \epsilon$ (for tiny $\epsilon > 0$), y is negative, heading to $-\infty$. Note that for $x = -\sqrt{1/2} - \epsilon$ (for tiny $\epsilon > 0$), y is positive, heading to $+\infty$.
As $x \to +\infty$, y goes up to 5/2. As $x \to -\infty$, y goes down to 5/2.
From these facts, we can sketch the plot. Check yourself by typing

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plot (13-9x+5x^2)/(2x^2-1)
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into wolframalpha.com, and you will see the plot.

(You will see that the function has a local minimum for a value of $x > \sqrt{1/2}$; now that we have studied derivatives, we know why! Find the critical points.)

8. The displacement (in feet) of a car moving in a straight line is given by $s(t) = 4t^2 + t + 1$, where t is measured in seconds.

(a). Find the average velocity of the car over the time interval [9,13].

The average velocity over the interval is

$$\frac{s(13) - s(9)}{13 - 9} = \frac{(4 \cdot 13^2 + 13 + 1) - (4 \cdot 9^2 - 9 - 1)}{13 - 9} = 89 ft/sec$$

(b). Find the instantaneous velocity of the car when $t = 10$. The instantaneous velocity at $t = 10$ is the limit as ϵ goes to zero of

$$\frac{s(10 + \epsilon) - s(10)}{10 + \epsilon - 10}$$

which is the derivative of $s(t)$ evaluated at $t = 10$: $s'(t) = 8t + 1$, so $s'(10) = 81$ ft/sec.

9. Find the equation of the tangent line to the curve $y - \sqrt{x} = 3$ ($x \geq 0$) at the point (4,5).

The curve is $y(x) = 3 + \sqrt{x}$. Since the slope is given by the derivative function, $y'(x) = (1/2)x^{-1/2} = \frac{1}{2\sqrt{x}}$, we know that the slope of the tangent line at (4,5) is $y'(4) = 1/4$. Thus, the equation of the tangent line at (4,5) has the form $y = (1/4)x + b$, and we can find b using the fact that the line must pass through the point (4,5): $5 = (1/4) \cdot 4 + b$, implying that $b = 4$. Thus, the equation of the tangent line at (4,5) is $y = (1/4)x + 4$.

10. If $h(b) = x^2 b^3 + 3b^2 - \frac{x^2}{b^3}$, find $h'(1)$. Also find $h''(x)$. $h'(b) = x^2 \cdot 3b^2 + 6b - x^2 \cdot (-3)b^{-4} = 3x^2 b^2 + 6b + \frac{3x^2}{b^4}$. Taking the second derivative, we get $h''(b) = 6x^2 b + 6 + 3x^2 \cdot (-4)b^{-5} = 6x^2 b + 6 - \frac{12x^2}{b^5}$.

Thus, $h'(1) = 3x^2 + 6 + 3x^2 = 6x^2 + 6$. Also, $h''(x) = 6x^3 + 6 - \frac{12}{x^3}$.

11. The graph of $f'(x)$ is shown below. At which of the marked points is (a). $f(x)$ the least? (b). $f(x)$ the greatest? (c). $f'(x)$ the least? (d). $f'(x)$ the greatest? (e). $f''(x)$ the least? (f). $f''(x)$ the greatest?

(a), (b). Since $f'(x) > 0$ over the interval of x-coordinates of interest (spanning all four points), we know that $f(x)$ is *increasing* over the interval. Thus, the least value of $f(x)$ occurs at A and the greatest occurs at D.

(c), (d). Among the four points marked, $f'(x)$ is greatest at D and is least at C.

(e), (f). Among the four points marked, the slope, $f''(x)$, of $f'(x)$ is greatest at C (where the slope is positive) and is least at B (where the slope is negative).

12. At a time t after it is thrown up in the air, a ball is at a height of $f(t) = ct^2 + 8t + 3$ meters, where c is a constant. We are told that at time $t = 1$ the ball is accelerating *downwards* at 6 meters/sec^2.

(a). What is the velocity of the ball at time $t = 1$? Is the ball going up or going down at time $t = 1$? The velocity at time t is given by $f'(t) = 2ct + 8$.

The acceleration at time t is given by $f''(t) = 2c$ (it is constant).

Since we are told that at time $t = 1$, the ball is accelerating downwards at 6 meters/sec^2, we know that $f''(1) = -6$, so $c = -3$.

We are asked for $f'(1) = 2(-3)(1) + 8 = 2$. Since $f'(1) > 0$, we know that the ball is going *up* at $t = 1$.

(b). How high does the ball go?

The ball reaches it highest point when the velocity is 0, i.e., when $f'(t) = 0$, or $-6t + 8 = 0$, which means that $t = 4/3$. The height at time $t = 4/3$ is $f(4/3) = -3 \cdot (4/3)^2 + 8 \cdot (4/3) + 3 = 25/3$ meters.

13. (a). Let $f(x) = \frac{x^2 e^{-x}}{2x^3 + e^x} - \ln(\sqrt{x})$. Find $f'(x)$. For the first term, we apply the quotient rule, and for the second we apply the chain rule:

$$f'(x) = \frac{(2x^3 + e^x)(x^2 e^{-x}(-1) + 2xe^{-x}) - x^2 e^{-x}(6x^2 + e^x)}{(2x^3 + e^x)^2} - \frac{1}{\sqrt{x}}(1/2)x^{-1/2}$$

(b). Let $f(x) = \cos(3\tan(3x))$. Find $f'(x)$.

We aply the chain rule:

$$f'(x) = (-\sin(3\tan(3x))(3\sec^2(3x))3$$

14. Suppose $xy - y^2 = e^{x^2}$. Compute $\frac{dy}{dx}$. We differentiate implicitly, on both side of the equation, with respect to x:

$$(xy' + 1 \cdot y) - 2yy' = e^{x^2} \cdot 2x,$$

which implies that

$$y' = \frac{e^{x^2}2x - y}{x - 2y}$$

15. (a). Suppose that $h(x) = 2x + g(x^2)$ and that $g'(u) = 3u^2$. Find $h'(x)$.

We take the derivative with respect to x, applying the chain rule to the "$g(x^2)$" term:

$$h'(x) = 2 + g'(x^2) \cdot 2x = 2 + 3(x^2)^2 \cdot 2x = 2 + 6x^5.$$

(b). Let $f(x) = 3e^{2x+1}$. Find the 100th derivative, $f^{(100)}(x)$.

First we compute $f'(x) = 3e^{2x+1} \cdot 2 = 3 \cdot 2 \cdot e^{2x+1}$.

Then we compute $f''(x) = 3 \cdot 2^2 e^{2x+1}$, and then $f'''(x) = 3 \cdot 2^3 e^{2x+1}$. Each time we take the derivative again, we pick up another factor of "2", from the chain rule (the derivative of "$2x + 1$").

Thus, $f^{(100)}(x) = 3 \cdot 2^{100} e^{2x+1}$.

16. The average cost per item, C, in dollars, of manufacturing a quantity q of cell phones is given by $C = (a/q) + b$, where a, b, are positive constants. (a). Find the rate of change of C as q increases. What are its units? (b). If production increases at a rate of 100 cell phones per week, how fast is the average cost changing? Is the average cost increasing or decreasing?

(a). Since $C(q) = (a/q)+b$, we get $C'(q) = -(a/q^2)$ dollars per cell phone is the rate of change of C as q increases.

(b). Now, considering q to be a function of t, $q(t)$, we can compute the derivative of C with respect to t:

$$\frac{dC}{dt} = -\frac{a}{q^2}\frac{dq}{dt} = -\frac{a}{q^2} \cdot 100,$$

in units of dollars per week (since we are told that $\frac{dq}{dt}$ is 100 cell phones per week – the production is increasing at the rate of 100 cell phones per week). Since dC/dt is negative (since a. q^2 are positive), we see that the average cost is decreasing.

17. (a). Compute $\lim_{t\to+\infty} 12te^{1/t} - 20t$. Rewriting the expression as a ratio, we get:

$$\lim_{t\to+\infty} 12te^{1/t} - 20t = \lim_{t\to+\infty} \frac{12e^{1/t} - 20}{1/t},$$

which has the form $\frac{-8}{0}$, so the limit does not exist (one could say that the limit is $-\infty$).

(a'). **This is the version that was intended:** Compute $\lim_{t\to+\infty} 12te^{1/t} - 12t$. Rewriting the expression as a ratio (whose limit form is $\frac{0}{0}$ as $x \to \infty$), and then applying l'Hopital's rule, we get:

$$\lim_{t\to+\infty} 12te^{1/t} - 12t = \lim_{t\to+\infty} \frac{12e^{1/t} - 12}{1/t},$$

which has the form $\frac{0}{0}$, so we apply l'Hopital's rule to get:

$$\lim_{t\to+\infty} \frac{12e^{1/t} - 12}{1/t} = \lim_{t\to+\infty} \frac{12e^{1/t}(-1/t^2)}{(-1/t^2)} = \lim_{t\to+\infty} 12e^{1/t} = 12.$$

(b). Compute $\lim_{t\to 0} \frac{1}{t} - \frac{1}{\sin t}$. Rewriting the expression as a ratio (whose limit form is $\frac{0}{0}$ as $x \to \infty$), and then applying l'Hopital's rule, we get:

$$\lim_{t\to 0}\frac{1}{t}-\frac{1}{\sin t}=\lim_{t\to 0}\frac{\sin t - t}{t\sin t}=\lim_{t\to 0}\frac{\cos t-1}{t\cos t+\sin t},$$

which still has the form $\frac{0}{0}$, so we apply l'Hopital's rule again to get

$$\lim_{t\to 0}\frac{\cos t-1}{t\cos t+\sin t}=\lim_{t\to 0}\frac{-\sin t}{t(-\sin t)+\cos t+\cos t}=\frac{0}{2}=0.$$

18. A rectangular beam is cut from a cylindrical log of radius 30 cm. The strength of a beam of width w and height h is proportional to wh^2. Find the width and height of the beam of maximum strength.

Draw a picture! (This is a problem from the textbook, so you can see the picture there.)
Then we see that $(w/2)^2+(h/2)^2=30^2$, so $h^2=4(900-w^2/4)=3600-w^2$.
Now, the strength, $f(w)$, is given by $f(w)=wh^2=w(3600-w^2)$.
We compute the derivative: $f'(w)=3600-3w^2$. Setting $f'(w)=0$, we get $3600=3w^2$, so $w=\sqrt{1200}$ and $w=-\sqrt{1200}$ are critical points. (Only $w=\sqrt{1200}$ is physically meaningful.) We can do the second derivative test: $f''(w)=-6w$, so $f''(\sqrt{1200})<0$, and we see that $w=\sqrt{1200}$ gives a local max, which is in fact a global max for $w\in(0,\infty)$ (since $f(0)=0$ is not greater, and $\lim_{w\to\infty}f(w)=-\infty$ is much less than $f(\sqrt{1200})$).
When $w=\sqrt{1200}$, $h=\sqrt{3600-(\sqrt{1200})^2}=\sqrt{2400}$ is the corresponding height.

19. Consider the function $f(x)=3x^5-5x^3$.
(a). Determine the critical points and classify each as a local max, local min, or neither. We compute $f'(x)=$

$15x^4-15x^2$. Thus, the critical points are given by solving $15x^4-15x^2=0$, which gives $15x^2(x^2-1)=0$, so $x=0,-1,1$ are the critical values of x.
In order to classify the critical points, we can apply the first derivative test: (i). $f'(x)$ does not change sign at $x=0$ (it is negative for $x=-0.01$ and also for $x=0.01$), so $x=0$ is not a local min or local max; (ii). $f'(x)$ changes from positive to negative at $x=-1$ ($f'(-1.01)>0$, while $f'(-.99)<0$), so $x=-1$ is a local max; (iii). $f'(x)$ changes from negative to positive at $x=1$ ($f'(0.99)<0$, while $f'(1.01)>0$), so $x=1$ is a local min.
(Alternatively, we can apply the second derivative test, using $f''(x)=60x^3-30x$: (i). $f''(0)=0$, so we cannot tell (use first derivative test instead); (ii). $f''(-1)=-30<0$, so $x=-1$ is a local max; (iii). $f''(1)=30>0$, so $x=1$ is a local min.)

(b). On which intervals is the function concave up? Concave down? Identify any inflection points.

Setting $f''(x)=60x^3-30x$ to 0, we get $x=0$ or $x=\sqrt{1/2}$ or $x=-\sqrt{1/2}$. Thus, we consider the corresponding intervals $-\infty,-\sqrt{1/2})$, $(-\sqrt{1/2},0)$, $(0,\sqrt{1/2})$, and $(\sqrt{1/2},\infty)$.
We can check that $f''(x)$ is positive on the interval $(-\infty,-\sqrt{1/2})$, where f is concave down.
We can check that $f''(x)$ is negative on the interval $(-\sqrt{1/2},0)$, where f is concave up. At $x=-\sqrt{1/2}$, there is an inflection point.
We can check that $f''(x)$ is positive on the interval $(0,\sqrt{1/2})$, where f is concave down. At $x=0$, there is an inflection point.
We can check that $f''(x)$ is negative on the interval $(\sqrt{1/2},\infty)$, where f is concave up. At $x=\sqrt{1/2}$, there is an inflection point.

(c). Find the global max/min on the interval $x\in[-10,10]$. The critical points $(x=0,-1,1)$ lie in the interval

[-10,10]; there is a local max at $x=-1$, where $f(-1)=-3+5=2$, and there is a local min at $x=1$, where $f(1)=3-5=-2$.
We must also check the endpoints of the interval: $f(-10)=-300,000+5000=-295,000$ and $f(10)=300,000-5000=295,000$.
Thus, the global max on the interval is at $x=10$ ($f(10)=295,000$), and the global min on the interval is at $x=-10$ ($f(-10)=-295,000$).

20. Joe runs a marathon. He gets more and more tired, so his speed decreases over time. His speed at certain times is given below.

Time (min)	0	15	30	45	60	75	90
Speed (mph)	12	12	11	10	9	8	0

Give upper and lower bounds on the distance that Joe runs during the first hour.

Let $v(t)$ be the velocity at time t. Since $v(t)$ is a decreasing function, we know that the left-hand-sum is an upper bound on total distance traveled, and the right-hand-sum is a lower bound on total distance traveled.

We are interested in the first hour, $t \in (0,1)$, so the left-hand-sum is given by

$$\frac{1}{4}(12+12+11+10) = 10.25,$$

and the right-hand-sum is given by

$$\frac{1}{4}(12+11+10+9) = 10.5,$$

so we know that the distance d travelled in the first hour obeys

$$10.25 \leq d \leq 10.5$$

(Note that if we knew the actual function $v(t)$, then the exact distance traveled in the first hour is given by $\int_0^1 v(t)dt$.)