Applied Calculus I
Practice Problems for Quiz # 3 – Solution Notes

1. Evaluate the following expressions:
(a). $\cos(\tan^{-1}(\frac{5}{9})) =$

Draw a right triangle and let θ be one of the two angles that is not $\pi/2$. Then, $\theta = \tan^{-1}(\frac{5}{9})$ if the side opposite θ is of length 5 and the side adjacent is of length 9. Then, the hypotenuse is of length $\sqrt{5^2+9^2} = \sqrt{106}$. Thus, $\cos(\tan^{-1}(\frac{5}{9})) = \cos\theta = \frac{9}{\sqrt{106}}$.

(b). $\tan(\sin^{-1}(\frac{\sqrt{3}}{2})) =$

Draw a right triangle and let θ be one of the two angles that is not $\pi/2$. Then, $\theta = \sin^{-1}(\frac{\sqrt{3}}{2})$ if the side opposite θ is of length $\sqrt{3}$ and the hypotenuse is of length 2. Then, the side adjacent is of length $\sqrt{2^2-(\sqrt{3})^2} = 1$. Thus, $\tan(\sin^{-1}(\frac{\sqrt{3}}{2})) = \tan\theta = \sqrt{3}$. (Note that $\theta = \pi/3$, 60 degrees.)

(c). $\sin(\frac{20\pi}{3}) =$

Note that $\frac{20\pi}{3} = 2\pi + 2\pi + 2\pi + \frac{2\pi}{3}$ (i.e., it is 3 revolutions, plus an additional angle $2\pi/3$ (120 degrees). Thus, $\sin(\frac{20\pi}{3}) = \sin(2\pi/3) = \frac{\sqrt{3}}{2}$.

(d). $\sec(\cos^{-1}(\frac{2}{3})) =$

Draw a right triangle and let θ be one of the two angles that is not $\pi/2$. Then, $\theta = \cos^{-1}(\frac{2}{3})$ if the side adjacent to θ is of length 2 and the hypotenuse is of length 3. Then, the side opposite is of length $\sqrt{3^2-2^2} = \sqrt{5}$. Thus, $\sec(\cos^{-1}(\frac{2}{3})) = \sec\theta = 1/(\cos\theta) = 1/(2/3) = 3/2$.

2. The intensity of radiation, I, oscillates sinusoidally between a low of 100 and a high of 1200. The time between one peak and the next peak is 5 minutes. At time 30 seconds after noon today, the radiation level is measured to be 650, and it is rising. Write the function $I(t)$ for intensity as a function of t, the number of seconds after noon today.

Since $I(t)$ oscillates between 100 and 1200, we know that the amplitude is (1200-100)/2=550. The baseline of the oscillation occurs at $I = 100 + 550 = 650$; i.e., it is a sin/cos shifted upwards by 650.
We are told that the period is 300 seconds (5 minutes, but t is measured in seconds).
We are told that $I(30) = 650$.
Draw a picture! We can view $I(t)$ as a sin function that is shifted right by 30. Thus,

$$I(t) = 650 + 550\sin(\frac{2\pi}{300}(t-30))$$

3. Find the asymptotes for the following function:

$$y = \frac{2x^2+11x+6}{x+18}$$

(a). Vertical: The denominator vanishes at $x = -18$, so this is a vertical asymptote.
(b). Horizontal: As $x \to +\infty$, y goes up to $+\infty$. As $x \to -\infty$, y goes down to $-\infty$. There is no horizontal asymptote.
(There is however a linear that is an asymptote: We can write $y = \frac{2x^2+11x+6}{x+18} = 2x - 25 + \frac{456}{x+18}$, from which we see that for values of x close to $+\infty$ or $-\infty$, the function behaves like the line $y = 2x - 25$.)
(c). Sketch a plot of $y(x)$

Note that for $x = -18 + \epsilon$ (for tiny $\epsilon > 0$), y is positive, heading to $+\infty$. Note that for $x = -18 - \epsilon$ (for tiny $\epsilon > 0$), y is negative, heading to $-\infty$.
As $x \to +\infty$, y goes up to $+\infty$. As $x \to -\infty$, y goes down to $-\infty$.
From these facts, we can sketch the plot. Check yourself by typing

```
plot (2x^2+11x+6)/(x+18)
```

into wolframalpha.com, and you will see the plot.

(You will see the function decreases towards the line $y = 2x - 25$ for $x \to +\infty$ and increases towards the line $y = 2x - 25$ for $x \to -\infty$. Line $y = 2x - 25$ is a (non-horizontal, non-vertical) asymptote.)

4. Find the asymptotes for the following function:

$$y = \frac{x^2 - x + 6}{x^3 - 1}$$

(a). Vertical: The denominator vanishes at $x = 1$, so this is a vertical asymptote.

(b). Horizontal:

We can rewrite (by dividing numerator and denominator by x^3)

$$y = \frac{x^2 - x + 6}{x^3 - 1} = \frac{\frac{1}{x} - \frac{1}{x^2} + \frac{6}{x^3}}{1 - \frac{1}{x^3}}$$

from which we see the behavior as x go to infinity. As $x \to +\infty$, y goes down to 0. As $x \to -\infty$, y goes up to 0.

Thus, $y = 0$ is a horizontal asymptote.

(c). Sketch a plot of $y(x)$

Note that for $x = -1 + \epsilon$ (for tiny $\epsilon > 0$), y is positive, heading to $+\infty$. Note that for $x = -1 - \epsilon$ (for tiny $\epsilon > 0$), y is negative, heading to $-\infty$.

As $x \to +\infty$, y goes down to 0. As $x \to -\infty$, y goes up to 0.

From these facts, we can sketch the plot. Check yourself by typing

```
plot (x^2-x+6)/(x^3-1)
```

into wolframalpha.com, and you will see the plot.

(You will see that the function has a bit of a "wiggle" around $x = 0$; when we study derivatives, we will see why.)

5. Find the asymptotes for the following function:

$$y = \frac{x^2 - x + 1}{x^2 - 4}$$

(a). Vertical: The denominator, $x^2 - 4 = (x - 2)(x + 2)$, vanishes at $x = 2$ and at $x = -2$, so these two values of x define the two vertical asymptotes.

(b). Horizontal:

We can rewrite (by dividing numerator and denominator by x^2)

$$y = \frac{x^2 - x + 1}{x^2 - 4} = \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{1 - \frac{4}{x^2}}$$

from which we see the behavior as x go to infinity. As $x \to +\infty$, y goes up to 1. As $x \to -\infty$, y goes down to 1.

Thus, $y = 1$ is a horizontal asymptote.

(c). Sketch a plot of $y(x)$

Note that for $x = 2 + \epsilon$ (for tiny $\epsilon > 0$), y is positive, heading to $+\infty$. Note that for $x = 2 - \epsilon$ (for tiny $\epsilon > 0$), y is negative, heading to $-\infty$.

Note that for $x = -2 + \epsilon$ (for tiny $\epsilon > 0$), y is negative, heading to $-\infty$. Note that for $x = -2 - \epsilon$ (for tiny $\epsilon > 0$), y is positive, heading to $+\infty$.

As $x \to +\infty$, y goes up to 1. As $x \to -\infty$, y goes down to 1.

From these facts, we can sketch the plot. Check yourself by typing

```
plot (x^2-x+1)/(x^2-4)
```

into wolframalpha.com, and you will see the plot.

(You will see that the function has a local minimum for a value of $x > 2$; when we study derivatives, we will see why.)

6. For what values of the constants b and c is the function f continuous on $(-\infty, \infty)$, where

$$f(x) = \begin{cases} cx + b & \text{if } x \in (-\infty, 2] \\ b + x & \text{if } x \in (2, 5) \\ cx^2 & \text{if } x \in [5, \infty) \end{cases}$$

The function is continuous for values of $x \neq 2, 5$.
In order to be continuous at $x = 2$, we need $c \cdot 2 + b = b + 2$, which implies that $c = 1$.
In order to be continuous at $x = 5$, we need $b + 5 = c \cdot 5^2 = 1 \cdot 5^2$, which implies that $b = 20$.

7. The height (in meters) of a particle moving along a vertical line is given by $h(t) = 100t^2 - 50$, where t is measured in seconds.

(a). Find the average velocity of the particle over the time interval [2,5].

The average velocity over the interval is

$$\frac{h(5) - h(2)}{5 - 2} = \frac{(2500 - 50) - (400 - 50)}{3} = 700m/sec$$

(b). Find the instantaneous velocity of the particle when $t = 4$.

The instantaneous velocity at $t = 4$ is the limit as ϵ goes to zero of

$$\frac{h(4 + \epsilon) - h(4)}{4 + \epsilon - 4}$$

which is approximately

$$\frac{h(4.001) - h(4)}{4.001 - 4} \approx 800.1$$

so we see that the instantaneous velocity is 800 m/sec.

(We can compute the derivative, $h'(t) = 200t$, and get the answer exactly as $h'(4) = 200 \cdot 4 = 800$ m/sec.)

8. Evaluate the limit

$$\lim_{x \to -1} \frac{x^2 + 3x + 2}{x + 1}$$

We rewrite,

$$\frac{x^2 + 3x + 2}{x + 1} = \frac{(x + 1)(x + 2)}{x + 1} = x + 2$$

from which we see that the limit as $x \to -1$ is $-1 + 2 = 1$.

9. Let

$$f(x) = \begin{cases} x + 4 & \text{if } x \leq -2 \\ 2x^2 + 3 & \text{if } x > -2 \end{cases}$$

Sketch the plot of $f(x)$ and determine the following limits, if they exist:
(a). $\lim_{x \to -2^-} f(x) = 2$
(b). $\lim_{x \to -2^+} f(x) = 11$
(c). $\lim_{x \to -2} f(x)$ does not exist

The plot is the line $x + 4$ to the left of $x = -2$ (and at $x = -2$), and is the parabola $2x^2 + 3$ to the right of $x = -2$. (You can plot each of these curves using wolframalpha.com.)