Applied Calculus I
Practice Problems for Quiz # 4 – Solution Notes

1. Find the equation of the tangent line to the curve $y = 4x^3 + 1$ at the point (2,33).

Since the slope is given by the derivative function, $y'(x) = 4 \cdot 3x^2 = 12x^2$, we know that the slope of the tangent line at (2,33) is $y'(2) = 48$. Thus, the equation of the tangent line at (2,33) has the form $y = 48x + b$, and we can find b using the fact that the line must pass through the point (2,33): $33 = 48 \cdot 2 + b$, implying that $b = -63$. Thus, the equation of the tangent line at (2,33) is $y = 48x - 63$.

2. Find the equation of the tangent line to the curve $y = 2x^2 - 5x + 3$ at the point where $x = 3$.

Since the slope is given by the derivative function, $y'(x) = 2 \cdot 2x - 5 = 4x - 5$, we know that the slope of the tangent line at the point where $x = 3$ (namely, the point $(3, 2 \cdot 3^2 - 5 \cdot 3 + 3) = (3, 6)$ is $y'(3) = 7$. Thus, the equation of the tangent line at (3,6) has the form $y = 7x + b$, and we can find b using the fact that the line must pass through the point (3,6): $6 = 7 \cdot 3 + b$, implying that $b = -15$. Thus, the equation of the tangent line at (3,6) is $y = 7x - 15$.

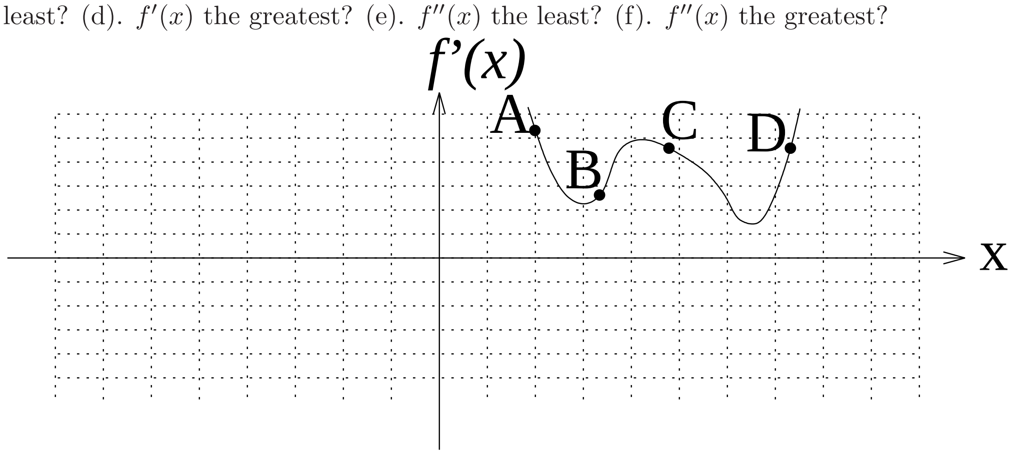
3. Evaluate the limit

$$\lim_{x\to\infty} \frac{5x^3 - 2x^2 + 7}{2x - 4x^2 - 7x^3}$$

$$\frac{5x^3 - 2x^2 + 7}{2x - 4x^2 - 7x^3} = \frac{5x^3 - 2x^2 + 7}{2x - 4x^2 - 7x^3} \cdot \frac{1/x^3}{1/x^3} = \frac{5 - \frac{2}{x} + \frac{7}{x^3}}{\frac{2}{x^2} - \frac{4}{x} - 7}$$

Now, as x goes to infinity, we see that the limit is $5/(-7)$, since all other terms with x in the denominator vanish in the limit.

4. The graph of $f'(x)$ is shown below. At which of the marked points is (a). $f(x)$ the least? (b). $f(x)$ the greatest? (c). $f'(x)$ the least? (d). $f'(x)$ the greatest? (e). $f''(x)$ the least? (f). $f''(x)$ the greatest?



(i). Since $f'(x) > 0$ over the interval of x-coordinates of interest (spanning all four points), we know that $f(x)$ is increasing over the interval. Thus, the least value of $f(x)$ occurs at A and the greatest occurs at D.

Among the four points marked, $f'(x)$ is greatest at A and is least at B.

Among the four points marked, the slope, $f''(x)$, of $f'(x)$ is greatest at D (where the slope is positive and greater than it is at B, where the slope is also positive, but smaller) and is least at A (where the slope is negative, as it is also at C, but it is more negative at A). Thus, $f''(x)$ is greatest at D and is least at A.

(ii). **Now suppose that the plotted function is not $f'(x)$, but rather $f(x)$.**

Among the four points marked, $f(x)$ is greatest at A and is least at B.

Among the four points marked, the slope, $f'(x)$, is greatest at D (where the slope is positive and greater than it is at B, where the slope is also positive, but smaller) and is least at A (where the slope is negative, as it is also at C, but it is more negative at A).

Among the four points marked, the slope is increasing at points A, B, and D (i.e., the function is concave upwards at A, B, and D); thus, at these points, $f''(x) > 0$. The slope is increasing most rapidly at point B.

Among the four points marked, the slope is decreasing at point C (i.e., the function is concave downwards at C); thus, at this point, $f''(x) < 0$. Thus, $f''(x)$ is least at point C.

5. If $f(x) = \frac{-23}{x^4}$, find $f'(2)$. Also find $f''(1)$.

$f(x) = -23x^{-4}$, so, by the power rule, $f'(x) = -23 \cdot (-4)x^{-5} = \frac{92}{x^5}$. Thus, $f'(2) = \frac{92}{2^5}$.
$f'(x) = 92x^{-5}$, so, by the power rule, $f''(x) = 92 \cdot (-5)x^{-6} = \frac{-460}{x^6}$. Thus, $f''(1) = -460$.

6. If $h(b) = ab + b^2 - 23b^3$, find $h'(2)$. Also find $h''(a)$.

$h'(b) = a + 2b - 23 \cdot 3b^2 = a + 2b - 69b^2$. Taking the second derivative, we get $h''(b) = 2 - 138b$.
Thus, $h'(2) = a + 2 \cdot 2 - 69 \cdot 2^2 = a - 272$. Also, $h''(a) = 2 - 138a$.

7. If $f(y) = 6y^2 - 12e^y + x^2$, find $f'(x)$ and $f''(a)$.

$f'(y) = 6 \cdot 2y - 12e^y = 12y - 12e^y$. (Note that the "$x^2$" term is a "constant" with respect to the variable y in $f(y)$, so the derivative with respect to the independent variable y is 0.)
Also, $f''(y) = 12 - 12e^y$.
Thus, $f'(x)$ is obtained by plugging in $y = x$ in $f'(y)$, so we get $f'(x) = 12x - 12e^x$.
Similarly, $f''(a)$ is obtained by plugging in $y = a$ in $f''(y)$, so we get $f''(a) = 12 - 12e^a$.

8. If $f(x) = (2x^2 - 3)(2x + 3)$, find $f'(x)$.

Multiply out to get $f(x) = 4x^3 + 6x^2 - 6x - 9$. Then, we see that $f'(x) = 12x^2 + 12x - 6$.

9. If $s(a) = \frac{-9}{x^3\sqrt{a}}$, find $s'(x)$.

We have that $s(a)$, a function of the independent variable a, is given by $s(a) = \frac{-9}{x^3}a^{-1/2}$. (Since a is the independent variable, the number x is a "constant".) Thus, $s'(a) = \frac{-9}{x^3} \cdot (-1/2)a^{-3/2} = \frac{9}{2x^3}a^{-3/2}$.
Thus, $s'(x)$ is given by plugging in x for a in $s'(a)$, yielding $s'(x) = \frac{9}{2x^3}x^{-3/2} = \frac{9}{2x^{9/2}}$.

10. Let $g(x) = 6x^5\sqrt{x} + \frac{3}{x^6\sqrt{x}}$. Find $g'(x)$.

Since $g(x) = 6x^{5.5} + 3x^{-6.5}$, we get that $g'(x) = 6 \cdot 5.5x^{4.5} + 3 \cdot (-6.5)x^{-7.5}$.

11. Let $f(x) = \sqrt{3x} + 7 + \frac{4}{x} + \frac{1}{3x^2}$. Find $f'(x)$.

Since $f(x) = \sqrt{3} \cdot x^{1/2} + 7 + 4x^{-1} + (1/3)x^{-2}$, we get that $f'(x) = (1/2)\sqrt{3} \cdot x^{-1/2} + 4 \cdot (-1)x^{-2} + (1/3) \cdot (-2)x^{-3}$.

12. A car moves along a straight road and its position at time t is given by $s(t) = 2t^3 - 20t^2 + 100t$. Find the velocity of the car at time $t = 0$. At which times is the car at rest? What is the total distance travelled by the car between time $t = 0$ and $t = 20$?

The velocity of the car at time t is $s'(t) = 6t^2 - 40t + 100$. Thus, the velocity at time $t = 0$ is $s'(0) = 100$.
The car is at rest when $s'(t) = 0$, i.e., when $6t^2 - 40t + 100 = 0$, which occurs when $t = \frac{40 \pm \sqrt{1600 - 4 \cdot 6 \cdot 100}}{2 \cdot 6}$. Since the expression within the square root is *negative*, there are *no* real number t for which $s'(t) = 0$; thus, the car is *never* at rest.
Since the velocity is always positive (we just saw that $s'(t) = 0$ has no real roots, and we saw that $s'(0) = 100 > 0$, so $s'(t)$ is always positive), we know the car is always moving in the $+s$ direction (it never switches direction). Thus, the total distance it travels between $t = 0$ and $t = 20$ is simply $s(20) - s(0) = 2 \cdot 20^3 - 20 \cdot 20^2 + 100 \cdot 20 - 0$.