AMS 151 (Fall, 2009) Joe Mitchell

Applied Calculus I
Practice Problems for Quiz # 5 – Solution Notes

1. Let $f(x) = \frac{1-x^3}{\sqrt{x+2^x}}$. Find $f'(x)$.

   We use the quotient rule:

$$f'(x) = \frac{\sqrt{x+2^x}\cdot(-3x^2) - (1-x^3)(1/2)(x+2^x)^{-1/2}(1+(\ln 2)2^x)}{x+2^x}$$

2. Let $f(x) = \sin x \cos x \tan 5x$. Find $f'(x)$.

   We use the product rule (twice): write $f(x) = (\sin x)\cdot(\cos x \tan 5x)$.

$$f'(x) = (\sin x)(\cos x(\sec^2 5x)\cdot 5 + \tan 5x(-\sin x)) + (\cos x \tan 5x)(\cos x)$$

3. Let $f(x) = e^{3x}\sin x \cos x$. Find $f'(x)$.

   We use the product rule (twice): write $f(x) = (e^{3x})\cdot(\sin x \cos x)$.

$$f'(x) = (e^{3x})(\sin x(-\sin x) + \cos x(\cos x)) + (\sin x \cos x)(e^{3x}\cdot 3)$$

4. Let $f(x) = (3x^4 - 2x^2 + 7)e^{3x}$. Find $f'(x)$ and $f''(x)$.

$$f'(x) = (3x^4 - 2x^2 + 7)(e^{3x}\cdot 3) + (e^{3x})(12x^3 - 4x) = (9x^4 + 12x^3 - 6x^2 - 4x + 21)e^{3x}$$

$$f''(x) = (9x^4 + 12x^3 - 6x^2 - 4x + 21)(e^{3x}\cdot 3) + (e^{3x})(36x^3 + 36x^2 - 12x - 4)$$

5. Let $f(x) = (3 + \sqrt{x^9\cdot 15^x})^{22}$. Find $f'(x)$.

$$f'(x) = 22(3+\sqrt{x^9\cdot 15^x})^{21}\cdot(1/2)(x^9\cdot 15^x)^{-1/2}\cdot(x^9\cdot(\ln 15)15^x + 15^x\cdot 9x^8)$$

6. Let $g(y) = 7^{(13y+22)}$. Find $g'(y)$.

$$g'(y) = (\ln 7)7^{(13y+22)}\cdot 13$$

7. Let $f(x) = \sqrt{x + \sqrt{x+\sqrt{x}}}$. Find $f'(x)$.

   We use the chain rule (in succession):

$$\begin{aligned} f(x) &= (1/2)(x+\sqrt{x+\sqrt{x}})^{-1/2}\cdot\frac{d}{dx}(x+\sqrt{x+\sqrt{x}}) \\ &= (1/2)(x+\sqrt{x+\sqrt{x}})^{-1/2}\cdot(1+(1/2)(x+\sqrt{x})^{-1/2}\cdot(1+(1/2)x^{-1/2})) \end{aligned}$$

8. Suppose that $h(b) = 2ab + g(b^2)$ and that $g'(w) = 3w^2$. Find $h'(b)$, $h'(a)$, and $h'(2)$.

   First we compute $h'(b)$, the derivative of the function h (which is a function of b; we treat a as constant).

$$h'(b) = 2a + g'(b^2)\cdot 2b.$$

Now, since $g'(w) = 3w^2$, we know that $g'(b^2) = 3(b^2)^2 = 3b^4$. Thus,

$$h'(b) = 2a + 3b^4 \cdot 2b = 2a + 6b^5.$$

This gives us $h'(b)$ for any b. Thus, we also get

$$h'(a) = 2a + 6a^5,$$

and

$$h'(2) = 2a + 6 \cdot 2^5.$$

9. Let $f(x) = \frac{e^{3x} \sin x}{\cos x}$. Find $f'(x)$.

$$f'(x) = \frac{(\cos x)(e^{3x} \cos x + (\sin x)(e^{3x} \cdot 3)) - (e^{3x} \sin x)(-\sin x)}{\cos^2 x}$$

10. Let $f(x) = \frac{6 \tan x + 9}{\sec x}$. Find $f'(x)$.

Method 1: Use the quotient rule:

$$f'(x) = \frac{(\sec x)(6 \sec^2 x) - (6 \tan x + 9)(\tan x \sec x)}{\sec^2 x},$$

where we have used the fact that

$$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = (-1)(\cos x)^{-2} \cdot (-\sin x) = \frac{\sin x}{\cos^2 x} = \tan x \sec x.$$

Method 2: Easier is to notice that $\sec x = 1/\cos x$ (by definition), so we can rewrite $f(x) = (\cos x)(6 \tan x + 9)$ and then apply the product rule:

$$f'(x) = (\cos x)(6 \sec^2 x) + (-\sin x)(6 \tan x + 9).$$

11. Let $h(x) = \sin(2x)$. Find the 83rd derivative, $h^{(83)}(x)$.

Begin by taking the first, second derivatives:

$$h'(x) = (\cos(2x)) \cdot 2 = 2 \cos(2x),$$

$$h''(x) = 2 \cdot (-\sin 2x) \cdot 2 = -2^2 \sin 2x,$$
$$h'''(x) = -2^2(\cos 2x) \cdot 2 = -2^3 \cos 2x,$$
$$h^{(4)}(x) = -2^3(-\sin 2x) \cdot 2 = 2^4 \sin 2x.$$

Thus, taking four derivatives "brings us back where we started" (from sin to cos (1st) to $-\sin$ (2nd) to $-\cos$ (3rd) to sin (4th)), except that we also pick up a factor of 2^4 (from the repeated chain rule). Thus,

$$h^{(8)}(x) = 2^8 \sin 2x, \quad h^{(12)}(x) = 2^{12} \sin 2x \quad \ldots \quad h^{(80)}(x) = 2^{80} \sin 2x.$$

Taking derivatives 3 more times, we get

$$h^{(81)}(x) = 2^{81} \cos 2x,$$
$$h^{(82)}(x) = -2^{82} \sin 2x,$$
$$h^{(83)}(x) = -2^{83} \cos 2x.$$

12. Find the equation of the tangent line to the curve $y = 3 \cos x \sin x$ at the point $(\pi/4, 3/2)$.

Since the slope is given by the derivative function,

$$y'(x) = (3\cos x)(\cos x) + (\sin x)(-3\sin x) = 3\cos^2 x - 3\sin^2 x = 3\cos(2x),$$

we know that the slope of the tangent line at $(\pi/4, 3/2)$ is $y'(\pi/4) = 3\cos(\pi/2) = 0$. Thus, the equation of the tangent line at $(\pi/4, 3/2)$ has the form $y = 0x + b = b$, and we can find b using the fact that the line must pass through the point $(\pi/4, 3/2)$: $3/2 = b$, implying that $b = 3/2$. Thus, the equation of the tangent line at $(\pi/4, 3/2)$ is $y = 3/2$.

13. Find the equation of the tangent line to the curve $y = 2\sec x - 4\cos x$ at the point $(\pi/3, 2)$.

Since the slope is given by the derivative function,

$$y'(x) = 2\sec x \tan x - 4(-\sin x) = 2\sec x \tan x + 4\sin x,$$

we know that the slope of the tangent line at $(\pi/3, 2)$ is $y'(\pi/3) = 6\sqrt{3}$. In computing the derivative above, we used the derivative of the function sec,

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = (-1)(\cos x)^{-2} \cdot (-\sin x) = \frac{\sin x}{\cos^2 x} = \tan x \sec x.$$

Thus, the equation of the tangent line at $(\pi/3, 2)$ has the form $y = 6\sqrt{3} \cdot x + b$, and we can find b using the fact that the line must pass through the point $(\pi/3, 2)$: $2 = 6\sqrt{3} \cdot (\pi/3) + b$, implying that $b = 2 - 2\pi\sqrt{3}$. Thus, the equation of the tangent line at $(\pi/3, 2)$ is $y = 6\sqrt{3} \cdot x + (2 - 2\pi\sqrt{3})$.

14. Find the equation of the tangent line to the curve $y = 2\ln(x)$ at $x = 5$.

Since the slope is given by the derivative function, $y'(x) = 2/x$, we know that the slope of the tangent line at $x = 5$ is $y'(5) = 2/5$. Thus, the equation of the tangent line at $x = 5$ has the form $y = (2/5)x + b$, and we can find b using the fact that the line must pass through the point $(5, y(5)) = (5, 2\ln 5)$: $2\ln 5 = (2/5) \cdot 5 + b$, implying that $b = 2\ln 5 - 2$. Thus, the equation of the tangent line at $x = 5$ is $y = (2/5)x + 2\ln 5 - 2$.

15. Find the equation of the tangent line to the curve $xy^3 + xy = 2$ at the point (1,1).

We differentiate implicitly: $(x \cdot 3y^2y' + 1 \cdot y^3) + (x \cdot y' + 1 \cdot y) = 0$, implying that at (1,1), we have $(1 \cdot 3 \cdot 1^2y' + 1 \cdot 1^3) + (1 \cdot y' + 1 \cdot 1) = 0$, or $4y' + 2 = 0$, so $y' = -1/2$. Thus, we know that the slope of the tangent line at (1,1) is $y'(1) = -1/2$. Thus, the equation of the tangent line at (1,1) has the form $y = (-1/2)x + b$, and we can find b using the fact that the line must pass through the point (1,1): $1 = (-1/2) \cdot 1 + b$, implying that $b = 3/2$. Thus, the equation of the tangent line at (1,1) is $y = (-1/2)x + 3/2$.

16. Find the equation of the tangent line to the curve $xy^3 + xy = 12$ at the point (6,1).

We differentiate implicitly: $(x \cdot 3y^2y' + 1 \cdot y^3) + (x \cdot y' + 1 \cdot y) = 0$, implying that at (6,1), we have $(6 \cdot 3 \cdot 1^2y' + 1 \cdot 1^3) + (6 \cdot y' + 1 \cdot 1) = 0$, or $24y' + 2 = 0$, so $y' = -1/12$. Thus, we know that the slope of the tangent line at (6,1) is $y'(6) = -1/12$. Thus, the equation of the tangent line at (6,1) has the form $y = (-1/12)x + b$, and we can find b using the fact that the line must pass through the point (6,1): $1 = (-1/12) \cdot 6 + b$, implying that $b = 3/2$. Thus, the equation of the tangent line at (6,1) is $y = (-1/12)x + 3/2$.

17. The temperature, H, in degrees Fahrenheit, of a can of soda that is put into a refrigerator to cool is given as a function of time, t, in hours, by $H(t) = 10 + 55e^{-2t}$. Find the rate of change of the temperature of the soda in units of degrees Fahrenheit per minute.

We compute: $H'(t) = 55e^{-2t} \cdot (-2) = -110e^{-2t}$, which is in units of degrees Fahrenheit per hour (the units of t). But we were asked to find the rate of change in units of degrees Fahrenheit per MINUTE, so we multiply by (1 hour)/(60 minutes) to get $-\frac{110}{60}e^{-2t}$. (It is negative since the temperature is going down.)

18. Let $f(x) = \sin(\cos(\sin x))$. Find $f'(x)$.

$$f'(x) = \cos(\cos(\sin x)) \cdot (-\sin(\sin x)) \cdot \cos x.$$

19. Let $f(x) = \sin(\cos(x^3))$. Find $f'(x)$.

$$f'(x) = \cos(\cos(x^3)) \cdot (-\sin(x^3)) \cdot 3x^2.$$

20. Let $f(x) = 4\cos(6\ln(2x))$. Find $f'(x)$.

$$f'(x) = -4\sin(6\ln(2x)) \cdot 6\frac{1}{2x} \cdot 2.$$

21. Let $f(x) = 7\log_9(ex)$. Find $f'(x)$.

One way to do this is to change the base of the logarithm (do it!). We will do it instead from first principles:

$$f(x) = \log_9(ex)^7,$$

so we have

$$9^{f(x)} = e^7 \cdot x^7,$$

so, taking the derivative of both sides, we get

$$(\ln 9)9^{f(x)} \cdot f'(x) = e^7 \cdot 7x^6.$$

Thus,

$$f'(x) = \frac{e^7 \cdot 7x^6}{(\ln 9)9^{f(x)}} = \frac{e^7 \cdot 7x^6}{(\ln 9)e^7 \cdot x^7} = \frac{7}{x\ln 9}.$$

22. Let $f(x) = 3x^{3x}$. Find $f'(x)$.

Take the logarithm:

$$\ln f(x) = \ln(3x^{3x}) = \ln 3 + 3x\ln x.$$

Now, taking the derivative with respect to x, we get:

$$\frac{1}{f(x)} \cdot f'(x) = 0 + 3x \cdot \frac{1}{x} + 3 \cdot \ln x,$$

which implies that

$$f'(x) = (3 + 3\ln x)f(x) = (3 + 3\ln x)3x^{3x}.$$