Applied Calculus I
Practice Problems for Quiz # 6 – Solution Notes

1. *Gasoline is pouring into a cylindrical tank of radius 3 feet. When the depth of the gasoline is 4 feet, the depth is increasing at 0.2 ft/sec. How fast is the volume of gasoline changing at that instant? (give units!)*

Let y be the depth in feet of the gasoline in the tank; y is actually a function, $y(t)$, of time. The volume, $V(y)$, when the depth is y is $V(y) = \pi r^2 y = 9\pi y$ cubic feet. Taking the derivative with respect to time, we get

$$\frac{dV}{dt} = 9\pi \frac{dy}{dt}.$$

At the moment when $y = 4$ feet, we are told that $\frac{dy}{dt} = 0.2$ ft/sec, at which point we get $\frac{dV}{dt} = 9\pi(0.2) = 1.8\pi$ cubic feet per second.

2. *A voltage V across a resistance R generates a current of $I = V/R$. If a constant voltage of 9 volts is put across a resistance that is increasing at a rate of 0.2 ohms per second when the resistance is 5 ohms, at what rate is the current changing? (give units!)*

The voltage V is constant, but $R(t)$ and $I(t)$ are both functions of time. In particular, $I(t) = V/R(t)$, so, taking derivatives with respect to time t, we get

$$\frac{dI}{dt} = -\frac{V}{R^2}\frac{dR}{dt}.$$

Thus, since $V = 9$ volts, the rate of change of the current, $\frac{dI}{dt}$, when $R(t) = 5$ ohms and $\frac{dR}{dt} = 0.2$ ohm/sec is

$$\frac{dI}{dt} = -\frac{9volts}{5^2 ohm^2} \cdot (0.2)(ohm/sec) = -\frac{1.8}{25}(volt/ohm)/sec$$

The current is decreasing (since the rate of change is negative), and the units are (volt/ohm) per second. You may know that "volt/ohm" is an "ampere", so the units are amperes per second.

3. *Compute* $\lim_{t\to 0^+} \frac{1}{t} - \frac{1}{e^t-1}$.

First, we write the expression with a common denominator:

$$\lim_{t\to 0^+} \frac{1}{t} - \frac{1}{e^t-1} = \lim_{t\to 0^+} \frac{e^t-1-t}{t(e^t-1)}.$$

Now, we see that the limit has the indeterminate form $\frac{0}{0}$ as $t \to 0^+$. Thus, we can apply l'Hopital's rule to get

$$\lim_{t\to 0^+} \frac{e^t-1-t}{t(e^t-1)} = \lim_{t\to 0^+} \frac{e^t-1}{te^t+(e^t-1)},$$

which still has the indeterminate form $\frac{0}{0}$. So we apply l'Hopital's rule again to get

$$\lim_{t\to 0^+} \frac{e^t-1-t}{t(e^t-1)} = \lim_{t\to 0^+} \frac{e^t-1}{te^t+(e^t-1)} = \lim_{t\to 0^+} \frac{e^t}{te^t+e^t+e^t)} = \frac{1}{2}.$$

4. Compute $\lim_{x\to 1} \frac{\ln x}{x^2-1}$

Since $\lim_{x\to 1} \frac{\ln x}{x^2-1}$ has the indeterminate form $\frac{0}{0}$, we apply l'Hopital's rule:

$$\lim_{x\to 1} \frac{\ln x}{x^2-1} = \lim_{x\to 1} \frac{(1/x)}{2x} = \frac{1}{2}$$

5. Compute $\lim_{t\to \pi} \frac{\sin^2 t}{t-\pi}$.

Since $\lim_{t\to \pi} \frac{\sin^2 t}{t-\pi}$ has the indeterminate form $\frac{0}{0}$, we apply l'Hopital's rule:

$$\lim_{t\to \pi} \frac{\sin^2 t}{t-\pi} = \lim_{t\to \pi} \frac{2\sin t\cos t}{1} = \frac{2\cdot 0\cdot(-1)}{1} = 0.$$

6. Which function dominates as $x \to \infty$: $f(x) = 0.01x^3$ or $g(x) = 50x^2$?

Since

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{50x^2}{(0.01)x^3} = 0,$$

we see that f dominates g as $x \to \infty$.

7. Find the global max and global min of the function $f(x) = 6x^3 + 18x^2 - 54x + 4$ over the interval $[-4, 4]$. Also determine all local mins and local max's on that interval. Determine on which intervals of $(-\infty, \infty)$ the function is increasing/decreasing. Also, determine on which intervals the function is concave up/up and identify any points of inflection. Show your work!

We begin by taking the first derivative:

$$f'(x) = 18x^2 + 36x - 54 = 18(x - 1)(x + 3).$$

The critical points of f correspond to values of x where the first derivative vanishes, so we set $f'(x) = 0$ and solve for x: $18(x - 1)(x + 3) = 0$ implies that $x = 1$ or $x = -3$. Thus, the critical points are $(1, f(1)) = (1, -26)$ and $(-3, f(-3)) = (-3, 166)$.

Now let's check each critical point to see if it is a local min, a local max, or neither:

At $x = -3$, the slope ($f'(x)$) changes from positive (to the left) to negative (to the right); this is easy to check by putting in values of x into $f'(x)$. Thus, f has a local MAX at $x = -3$, at the point $(-3, 166)$.

At $x = 1$, the slope ($f'(x)$) changes from negative (to the left) to positive (to the right); this is easy to check by putting in values of x into $f'(x)$. Thus, f has a local MIN at $x = 1$, at the point $(1, -26)$.

What we know so far is that f is INCREASING on the interval $(-\infty, -3)$, has a critical point (zero slope) at $x = -3$, then is DECREASING on the interval $(-3, 1)$, has a critical point (zero slope) at $x = 1$, then is INCREASING on the interval $(1, \infty)$.

Next, in order to discover information about being concave up or concave down, and points of inflection, let's look at the second derivative:

$$f''(x) = 36x + 36.$$

The second derivative vanishes if $36x + 36 = 0$, i.e., if $x = -1$. Thus, the point $(-1, f(-1)) = (-1, 70)$ is a *candidate* inflection point. We check: to the left of $x = -1$, $f''(x) < 0$, and to the right of $x = -1$, $f''(x) > 0$. (Just plug in some values to check.) Thus, the function f is concave DOWN on the interval $(-\infty, -1)$, and is concave UP on the interval $(-1, \infty)$. Since there is a change in concavity at $x = -1$, the point $(-1, 70)$ is a point of inflection.

Now, returning to the analysis of $f(s)$ on the interval $[-4, 4]$, we see that the local MAX, $(-3, 166)$, lies in this interval, and also that the local MIN, $(1, -26)$, also lies in this interval. On the interval $[-4, 4]$, there can also be local min/mas at the endpoints of the interval, so we check these next. At $x = -4$, $f(-4) = 124$; at $x = 4$, $f(4) = 460$. Thus, within $[-4, 4]$, f has a local min at $x = -4$ and a local max at $x = 4$.

In order to determine a *global* max/min, we take the largest/smallest value among the local max/min's that lie in the interval.

There are two local max's in the interval: $(-3, 166)$ and $(4, 460)$; the GLOBAL MAX of f on $[-4, 4]$ occurs at $(4, 460)$.

There are two local min's in the interval: $(-4, 124)$ and $(1, -26)$; the GLOBAL MIN of f on $[-4, 4]$ occurs at $(1, -26)$.

You can use WolframAlpha to show a plot of f.

8. Find the global max and global min of the function $f(x) = x/(x^2 + 1)$ over the interval $[-0.5, 4]$. Also determine all local mins and local max's on that interval. Determine on which intervals of $(-\infty, \infty)$ the function is increasing/decreasing. Also, determine on which intervals the function is concave up/up and identify any points of inflection. Show your work!

To be provided.

9. Find the global max and global min of the function $f(x) = x^7(x + 5)^6$ over the interval $[-11, 12]$. Also determine all local mins and local max's on that interval. Determine on which intervals of $(-\infty, \infty)$ the function is increasing/decreasing. Also, determine on which intervals the function is concave up/up and identify any points of inflection. Show your work!

To be provided.

10. Find the point on the line $-7x + 6y + 4 = 0$ that is closest to the point (2,2).

To be provided.

11. A rectangle has one side on the x-axis and two vertices on the curve $y = 1/(1 + x^2)$. Find the vertices of the rectangle with maximum area.

To be provided.

12. If you have 100 feet of fencing and want to enclose a rectangular area up against a long, straight wall, what is the largest area you can enclose? We did this one in class.

13. A cylinder is inscribed in a right circular cone of height 2 and radius (at the base) equal to 4. Find the dimensions of such a cylinder that has maximum volume.

To be provided.