Applied Calculus I
Quiz # 6 – Solution Notes

1. (4 points) *Compute* $\lim_{t\to+\infty} 12te^{1/t} - 12t$.

We rewrite the expression:

$$\lim_{t\to+\infty} 12te^{1/t} - 12t = \lim_{t\to+\infty} \frac{12(e^{1/t}-1)}{1/t},$$

and then see that it is of the form $\frac{0}{0}$, so we apply l'Hopital's rule to get

$$\lim_{t\to+\infty} \frac{12(e^{1/t}-1)}{1/t} = \lim_{t\to+\infty} \frac{12(e^{1/t}(-1/t^2))}{-1/t^2} = \lim_{t\to+\infty} 12e^{1/t} = 12.$$

2. (4 points) *Compute* $\lim_{t\to0^+} \frac{1}{t} - \frac{1}{e^t-1}$.

We rewrite the expression by writing with a common denominator:

$$\lim_{t\to0^+} \frac{1}{t} - \frac{1}{e^t-1} = \lim_{t\to0^+} \frac{e^t-1-t}{t(e^t-1)},$$

and then see that it is of the form $\frac{0}{0}$, so we apply l'Hopital's rule to get

$$\lim_{t\to0^+} \frac{e^t-1-t}{t(e^t-1)} = \lim_{t\to0^+} \frac{e^t-1}{te^t+e^t-1},$$

and see that it is still of the form $\frac{0}{0}$, so we apply l'Hopital's rule again to get

$$\lim_{t\to0^+} \frac{e^t-1}{te^t+e^t-1} = \lim_{t\to0^+} \frac{e^t}{te^t+e^t+e^t} = \frac{1}{2}.$$

3. (4 points) *A cylinder is inscribed in a right circular cone of height 12 and radius (at the base) equal to 4. Find the dimensions (radius and height) of such a cylinder that has maximum possible volume.*

Look at a cross-section in the (x, y) plane (where y is the "vertical" direction along the axis of the circular cone): the cone intersects the plane in a line segment, which lies on the line $y = 12 - 3x$ (which crosses the y-axis at $y = 12$ and the x-axis at $x = 4$, since the radius of the base of the cone is 4 and the height of the cone is 12). In this cross-section, a maximal volume cylinder will be a rectangle, whose upper right corner, (x, y) lies on the line $y = 12 - 3x$. The cylinder has radius x and height y.

The volume, $V(x)$, of the cylinder is then $V(x) = \pi x^2(12 - 3x)$.

We want to maximize $V(x)$, for $0 \le x \le 4$.

We compute the derivative: $V'(x) = 24\pi x - 9\pi x^2$. Setting $V'(x) = 0$, we get $x = 0$ or $x = 8/3$. If $x = 0$, the cylinder has no volume at all, so that is not the radius of interest.

We use the second derivative test: $V''(x) = 24\pi - 18\pi x$, so $V''(8/3) = -24\pi < 0$, so we know that $x = 8/3$ is a local maximum.

Thus, we pick radius $x = 8/3$ and height $y = 12 - 3(8/3) = 4$.

4. *Consider the function* $f(x) = 2x^3 + 3x^2 - 36x + 6$.

(a). (4 points) *Determine the critical points and classify each as a local max, local min, or neither.*

We compute $f'(x) = 6x^2 + 6x - 366(x - 2)(x + 3)$; to find the critical points, we set $f'(x) = 0$ and solve: critical points occur at $x = -3$ and $x = 2$.

In order to classify them, we can apply the second derivative test: $f''(x) = 12x + 6$, so $f''(-3) = -30 < 0$, so $x = -3$ corresponds to a local max (with $f(-3) = 87$). $f''(2) = 30 > 0$, so $x = 2$ corresponds to a local min (with $f(2) = -38$).

(b) .(2 points) *On which intervals is the function concave up? Concave down? Identify any inflection points.* Since $f''(x) = 12x + 6$, we see that $f''(x) = 0$ if $x = -1/2$; thus, f is concave down on the interval $(-\infty, -1/2)$

(where $f''(x) < 0$), f has an inflection point at $x = -1/2$, and f is concave up on the interval $(-1/2, \infty)$ (where $f''(x) > 0$).

(c). (2 points) *Find the global max/min on the interval* $x \in [-10, 10]$.

We already found in (a) that f has a local min at $(2, -38)$ and a local max at $(-3, 87)$. We need to check the value of f also at the endpoints of the interval [-10,10]: $f(-10) = -1334$ and $f(10) = 1946$. Thus, the global maximum of f over [-10,10] occurs at $x = 10$ ($f(10) = 1946$), and the global minimum of f over [-10,10] occurs at $x = -10$ ($f(-10) = -1334$).