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COMPUTATIONAL GEOMETRY
Examples: Duality

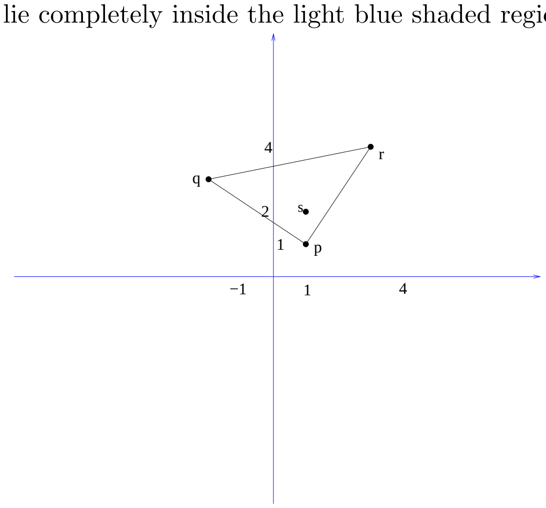
(1). *Describe what the dual is for a triangle, Δpqr. (The triangle is considered to be a closed region, consisting of all points inside or on the boundary of the triangle. Use the standard duality transform of the text (not polar duality).)*

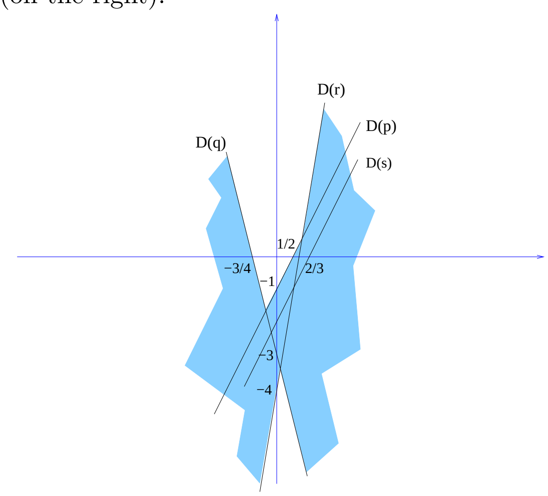
Suppose that, as in the example below, the points p, q, and r are ordered clockwise, with q leftmost, r rightmost. Then, p lies below the line through q and r.

Then, a point $s \in \Delta pqr$ lies below line qr, above line qp, and above line pr. Thus, the dual of s, $D(s)$, is a line that lies above the point $D(q) \cap D(r)$, and below the point $D(q) \cap D(p)$ and below the point $D(p) \cap D(r)$. Thus, the dual of the triangle is described by the set of all lines that lie above one point and below two others (or vice versa, depending on the original configuration of the triangle).

As an example, draw a picture to represent the dual of the triangle determined by the points $p = (1, 1)$, $q = (-2, 3)$, and $r = (3, 4)$. (Label your axes and make it clear what the important points are.)

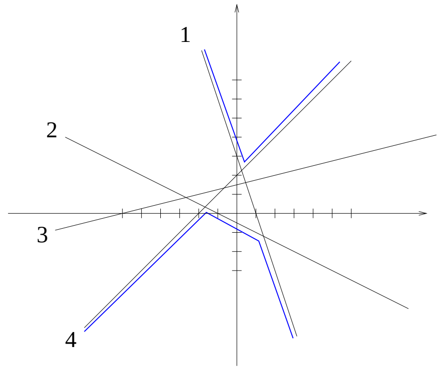
Below, I plot the points and their dual lines ($y = 2x - 1$, $y = -4x - 3$, and $y = 6x - 4$). I also show a point s inside the triangle pqr and its dual line, $D(s)$. The duals of the points in triangle pqr are the lines that lie completely inside the light blue shaded region (on the right).

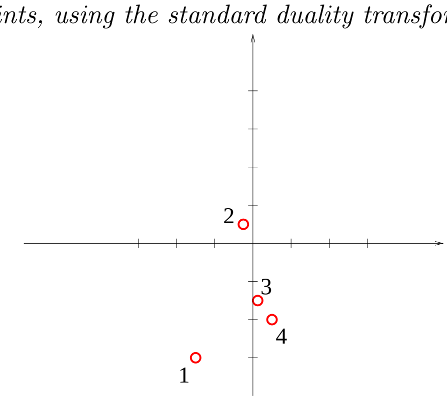




(2). *Let S be the set of points $\{p_1, \ldots, p_4\}$ = {(-3/2,-3), (-1/4,1/2), (1/8,-3/2), (1/2,-2)}.*

(a). Draw the arrangement of the duals to these points, using the standard duality transform of the text.



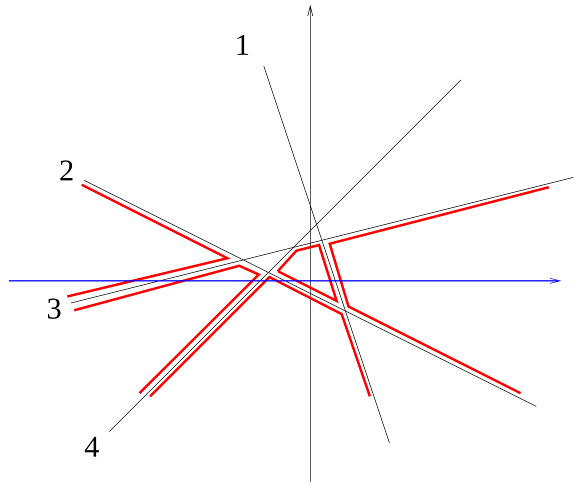


(b). Let ℓ be the x-axis. Highlight the edges of the cells of the zone of ℓ (color them red). How many are there? How does this number compare to the upper bound given by Theorem 6.2.2?

Below I show ℓ in blue and highlight with red segments the edges of the zone.

Starting at $y = -\infty$, line ℓ is in an unbounded cell of 2 sides, then enters cell of 3 sides, then a cell of 3 sides, then a cell of 4 sides, then a cell of 3 sides. In total, there are 15 edges of the 5 cells that make up the zone. So the complexity of the zone in this case is 15.

Since $n = 4$ here, the theorem tells us that there are at most $6n = 24$ edges in the zone. Obviously, $15 \leq 24$.



(c). Which lines ℓ_i correspond to points p_i that are vertices on the upper convex hull of S? Which lines ℓ_i correspond to points p_i that are vertices on the lower convex hull of S? What do you observe about the connection between the convex hull of S and the arrangement of dual lines?

The vertices, p_1, p_2, p_4, of the upper hull correspond to the lines ℓ_1, ℓ_2, ℓ_4 that define the *lower envelope* (highlighted in blue above) of the 4 dual lines (which occur in left-to-right order as ℓ_4, ℓ_2, ℓ_1). The lower hull vertices are p_1, p_4, which correspond to the lines that define the *upper envelope*, which occur in left-to-right order as ℓ_1, ℓ_4. In general, the upper (resp., lower) convex hull of the points corresponds to the lower (resp., upper) envelope of the lines in the dual.

(d). Suppose that points p_1-p_2 are "red" and p_3-p_4 are "blue". Give a ham sandwich cut of the points.

Many solutions are possible. We desire a line that simultaneously splits the red points in half and the blue points in half. Since there are only 4 points, this is essentially trivial.