

COMPUTATIONAL GEOMETRY

Homework Set # 7 – Solution Notes

(1). For each of the following sign vectors, draw an arrangement of 5 distinct lines, $\{\ell_1, \dots, \ell_5\}$, that has a face (either a 2-face (cell), a 1-face (edge), or a 0-face (vertex)) with the corresponding sign vector (where “+” means lying strictly above a line, “-” means strictly below, and “0” means on the line). Clearly label each line and highlight the face.

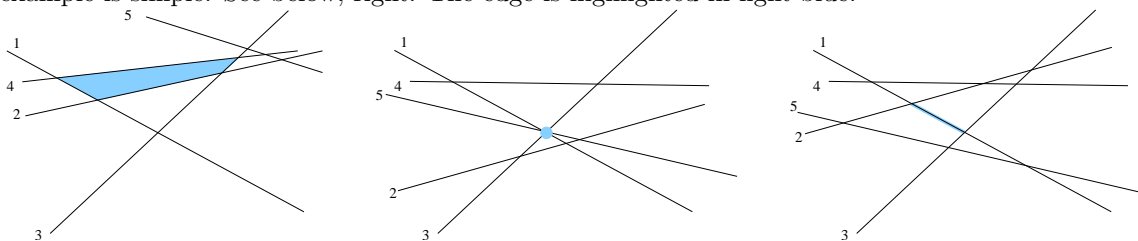
Give the dimension of each such face. Also, say for each case whether the arrangement is simple or not. (You should make your example simple if it is possible to do so.)

NOTE: Many different arrangements are possible to illustrate each case; chances are, your figure looks different from mine!

(a). $(+, +, +, -, -)$; This is a 2-face, since it lies interior to all 5 half-planes. The arrangement in the example is simple; see below, left. The 2-face is highlighted in light blue.

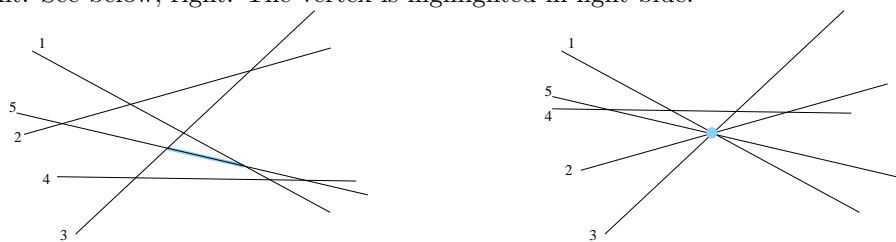
(b). $(0, +, 0, -, 0)$; This is a 0-face (vertex), since it lies on 2 or more lines. The arrangement cannot be simple, since this 0-face is common to 3 lines and in a simple arrangement no three lines can pass through a common point. See below, middle. The vertex is highlighted in light blue.

(c). $(0, -, +, -, +)$; This is a 1-face (edge), since it lies on exactly one line (one “0”). The arrangement in the example is simple. See below, right. The edge is highlighted in light blue.



(d). $(-, -, -, +, 0)$; This is a 1-face (edge), since it lies on exactly one line (one “0”). The arrangement in the example is simple. See below, left. The edge is highlighted in light blue.

(e). $(0, 0, 0, -, 0)$; This is a 0-face (vertex), since it lies on 2 or more lines. The arrangement cannot be simple, since this 0-face is common to 4 lines and in a simple arrangement no three lines can pass through a common point. See below, right. The vertex is highlighted in light blue.



(Optional question for extra thought: Is it possible to find one set of 5 lines whose arrangement contains faces with all five sign vectors?) NO, it is not possible. There are several ways to see this. One is to note that the existence of a face with sign vector $(0, 0, 0, -, 0)$ implies that lines $\ell_1, \ell_2, \ell_3, \ell_5$ must all pass through a common point (that vertex, v , with that sign vector); then, there can be no face with sign vector $(0, +, 0, -, 0)$, since such a face (0-face) lies at the intersection of ℓ_1, ℓ_3 , and ℓ_5 (because of the three 0's), meaning it lies at point $v = \ell_1 \cap \ell_2 \cap \ell_3 \cap \ell_5$, contradicting the “+” in the sign vector associated with ℓ_2 .

(2). Let L be a set of n distinct lines in the plane, no two of which are parallel. Assume that exactly k ($k \leq n$) of the lines (call this set L_0) all pass through a common point, p_0 . Assume that no three of the other $n - k$ lines (in the set $L_1 = L \setminus L_0$) pass through a common point, and no line of L_1 passes through p_0 . Assume also that no line of L_0 passes through an intersection point between two lines of L_1 . Derive formulas for the number of vertices, edges, and faces (cells) in the arrangement, $\mathcal{A}(L)$, of the lines L . Your formulas for v , e , and f will depend on n and k . (Note that your formulas should match what we get for

the case $k = 0$, when the set of lines forms a simple arrangement.) (This last sentence about verifying the case $k = 0$ is misleading and should be dropped.)

Let $m = n - k$ be the number of lines in L_1 . Let k be an arbitrary nonnegative integer, and consider various values for m : $m = 0, 1, 2, \dots$. We let $v(m, k)$, $e(m, k)$, $f(m, k)$ denote the number of vertices, edges, and faces for any given choice of m and k .

Just to get a feel for the problem, we start with some small cases.

If $m = 0$, we have k lines all passing through p_0 , so we have 1 vertex, $2k$ edges, and $2k$ faces. Thus, $v(0, k) = 1$, $e(0, k) = 2k$, and $f(0, k) = 2k$.

If $m = 1$, we have k lines all passing through p_0 and one other line that cuts across all k of the lines of L_0 , so we have a total of $k + 1$ vertices. Each line of L_0 has 2 vertices on it, implying 3 edges. The line of L_1 has k vertices on it, implying $k + 1$ edges. Thus, $v(1, k) = k + 1$, $e(1, k) = 3k + (k + 1)$, and $f(1, k) = 3k + 1$.

Now, more generally, each of the m lines in L_1 intersects all k lines of L_0 and all $m - 1$ other lines of L_1 ; thus, there are $k + m - 1$ vertices along each of the m lines of L_1 , implying $k + m$ edges along each. Each of the k lines of L_0 intersects the other $k - 1$ lines of L_0 (at the single vertex p_0) and all m lines of L_1 ; thus, there are $m + 1$ vertices along each of the k lines of L_0 , implying $m + 2$ edges along each.

Summing the number of edges along each line gives $m \cdot (k + m) + k \cdot (m + 2)$. Thus, the total number of edges is $e(m, k) = 2km + m^2 + 2k = 2k(n - k) + (n - k)^2 + 2k$. (Note that $e(m, 0) = n^2$, as we expect for a simple arrangement of n lines.)

The total number of vertices is obtained by counting all vertices defined by intersections among lines within L_0 (this is just 1, the point p_0), plus all vertices defined by intersections among lines within L_1 (this is just $\binom{m}{2}$, since every pair of lines intersects in a unique vertex), plus all vertices defined by intersections between a line of L_0 and a line of L_1 (this is just mk , since each pair crosses at a unique vertex). In total, there are $v(m, k) = \binom{m}{2} + mk + 1 = \binom{n-k}{2} + k(n - k) + 1$ vertices. (Here, we have been assuming that $k \geq 2$, since we assume the vertex p_0 exists. Thus, it is not relevant to consider $k = 0$ in this formula.)

By Euler's formula, we get $f = e - v' + 2$, where $v' = v + 1$ is the total number of vertices including the vertex at infinity (where all of the infinite rays connect). Thus, $f(m, k) = e(m, k) - v(m, k) + 1 = 2km + m^2 + 2k - \binom{m}{2} - mk - 1 + 1 = km + 2k + m(m + 1)/2 = k(n - k) + 2k + (n - k)(n - k + 1)/2$. (We have been assuming that $k \geq 2$ in our derivation, so it is not really relevant to verify this formula for the case $k = 0$.)

(3). *O'Rourke, problem 3(a), section 6.5.3, page 205.* Let $p = (a, b)$ be a point in the (a, b) -plane (the primal plane), and let $L : ax + by = 1$ be its polar dual, in the (x, y) -plane. Let C be the unit circle centered at the origin. (We place a copy of C in both the primal and dual planes.) Rewrite the equation of L : $y = (-a/b)x + (1/b)$. Then, it is clear that L is a line of slope $-a/b$, which means that it is orthogonal to the line, $\ell : y = (b/a)x$, that passes through the origin and (a, b) (since their slopes are negative reciprocals of each other).

Now, the distance from the origin to the point, $q = L \cap \ell$ (which is the closest point of L to the origin), can be computed by solving for q 's coordinates:

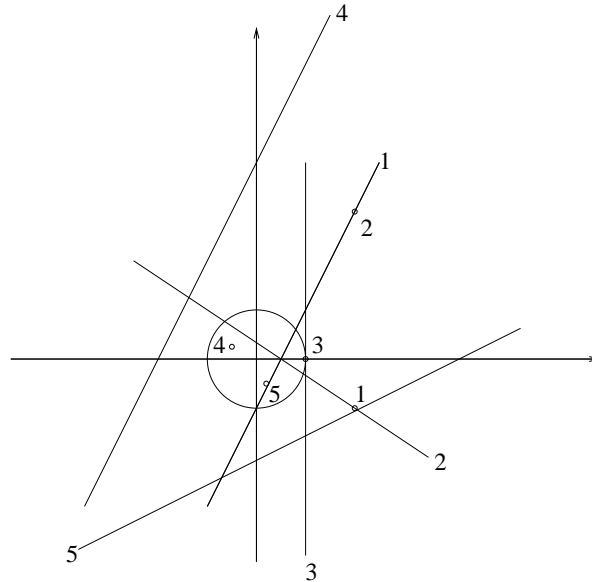
$$q = \left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2} \right),$$

which gives us the distance $d = 1/(a^2 + b^2)$ from the origin to L . (Alternatively, you can arrive at this formula easily using similar triangles.) We note that d is the reciprocal of the distance from the origin to $p = (a, b)$.

Thus, the dual line L is obtained by drawing a line L that is orthogonal to the line ℓ through p and the origin, at a distance from the origin that is the reciprocal of the distance from p to the origin. There are three cases:

- (1). p is interior to the circle C . Then, L is outside the circle C , since $d > 1$.
- (2). p is on the circle C . Then, L is tangent to the circle C , at the point p , since $d = 1$.
- (3). p is exterior to the circle C . Then, L intersects outside the circle C , since $d < 1$.

Draw the lines that are the polar duals of the points $(2,-1)$, $(2,3)$, $(1,0)$, $(-0.5,0.25)$, $(0.25,-0.5)$. Below, I draw the points and lines in the same diagram, labeling them “1”, “2”, “3”, “4”, “5”, in the order the points are given to us.



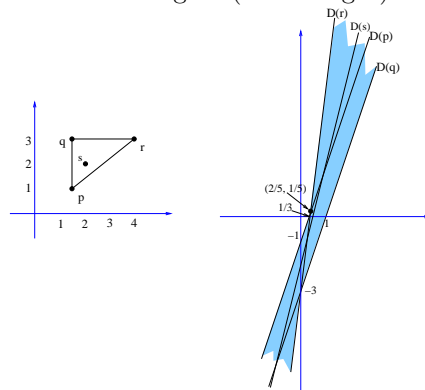
(4). Describe what the dual is for a right triangle, Δpqr , with pq vertical (parallel to the y -axis) and qr horizontal (parallel to the x -axis). (The triangle is considered to be a closed region, consisting of all points inside or on the boundary of the triangle. Use the standard duality transform of the text (not polar duality).)

Suppose that, as in the example below, the points p , q , and r are ordered clockwise, with p and q leftmost, r rightmost. Then, p lies below the line through q and r .

Then, a point $s \in \Delta pqr$ lies below line qr and above line pr and has x -coordinate greater than that of p and q . Thus, the dual of s , $D(s)$, is a line that lies above the point $D(q) \cap D(r)$ and below the point $D(p) \cap D(r)$ and has slope greater than $2 \cdot \frac{3}{2} = 3$. Thus, the dual of the triangle is described by the set of all lines that lie above one point and below another point, (or vice versa, depending on the original configuration of the triangle) with slope above a certain value.

As an example, draw a picture to represent the dual of the triangle determined by the points $p = (3/2, 1)$, $q = (3/2, 3)$, and $r = (4, 3)$. (Label your axes and make it clear what the important points are.)

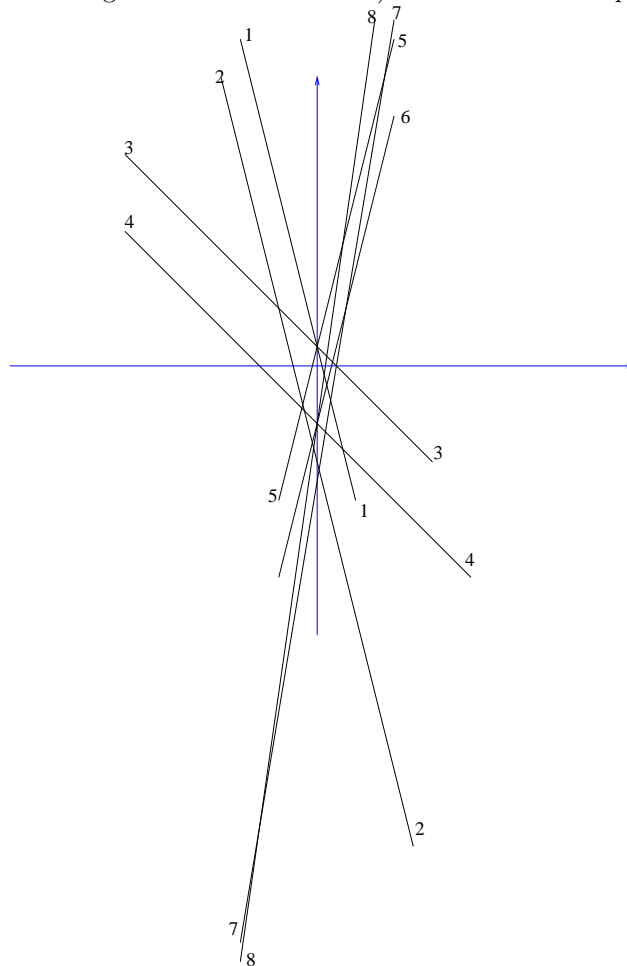
Below, I plot the points and their dual lines ($y = 2x - 1$, $y = -4x - 3$, and $y = 6x - 4$). I also show a point s inside the triangle pqr and its dual line, $D(s)$. The duals of the points in triangle pqr are the lines that lie completely inside the light blue shaded region (on the right).



(5). Let S be the set of points $\{p_1, \dots, p_8\} = \{(-4,-1), (-4,5), (-1,-1), (-1,3), (4,-1), (4,3), (6,6), (7,3)\}$. (You have seen these points in HW5 and HW6 already.)

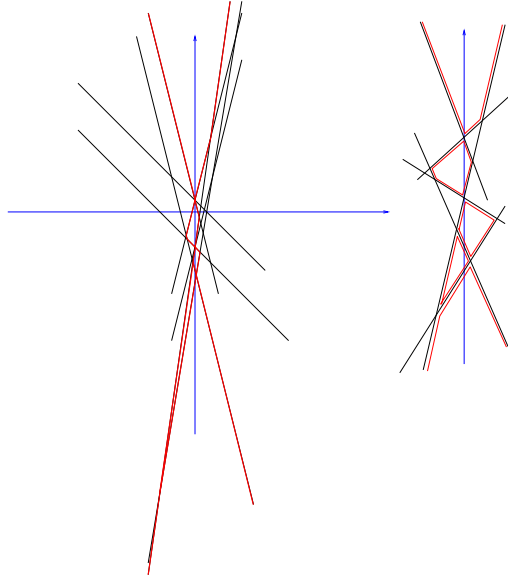
(a). Draw the arrangement of the duals to these points. (It is very useful to use graph paper or a computer drawing package for this. Make sure that you label the coordinate axes with important values of the coordinates.) Label the lines, letting ℓ_i denote the dual of point p_i .

I show the arrangement below. I have drawn the figure with the x -coordinates scaled by a factor of 2 (“stretching” the figure horizontally), in order to show the structure of the arrangement more clearly. (Scaling does not change the arrangement structure at all.) I label the dual of p_i with the number i .



(b). Let L be the y -axis. Highlight the edges of the cells of the zone of L (color them red). How many are there? How does this number compare to the bound given by Theorem 6.2.2?

In the figure below, I show the zone of L and highlight the edges with red segments. Starting at $y = -\infty$, line L is in an unbounded cell of 3 sides, then enters another cell with 3 sides, then a cell with 4 sides, which it exits at the vertex $(0,-3)$, where it enters a cell having 5 sides, which it exits at the vertex $(0,1)$, where it enters an unbounded cell having 3 sides. (Recall that in order for a cell to be in the zone of L , it must be intersected by L , and cells (2-faces) are *open* sets, not including their boundary edges and vertices; thus, cells that touch L at one of their vertices but are not intersected by L are *not* part of the zone of L .) Thus, the zone of L has complexity $3 + 3 + 4 + 5 + 3 = 18$. In comparison, the upper bound given by Theorem 6.2.2 is $6n = 6 \cdot 8 = 48$, which is much higher.



(c). Which lines ℓ_i correspond to points p_i that are vertices on the upper convex hull of S ? Which lines ℓ_i correspond to points p_i that are vertices on the lower convex hull of S ? What do you observe about the connection between the convex hull of S and the arrangement of dual lines?

The vertices, p_2, p_7, p_8 , of the upper hull correspond to the lines ℓ_2, ℓ_7, ℓ_8 that define the lower envelope of the 8 dual lines (which occur in left-to-right order as ℓ_8, ℓ_7, ℓ_2). The lower hull vertices are p_1, p_5, p_6 , which correspond to the lines that define the upper envelope, which occur in left-to-right order as ℓ_1, ℓ_5, ℓ_6 . In general, the upper (resp., lower) convex hull of the points corresponds to the lower (resp., upper) envelope of the lines in the dual.

(d). Suppose that points $p_1 - p_4$ are "red" and $p_5 - p_8$ are "blue". Give a ham sandwich cut of the points.

