Binary Space Partitions for Orthogonal Segments and Hyperrectangles

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Binary Space Partitions (BSP):
**BSP Trees:**

**Input:** set $S$ of $n$ disjoint objects in $\mathbb{R}^d$

**Output:** BSP: convex subdivision induced by hyperplanes such that each (open) cell intersects at most one object

**GOAL:** Small trees (few fragments)

size of BSP = total number of object fragments (leaves)

**autopartition** – splitting hyperplanes are supporting planes of (polyhedral) objects $S$

“perfect BSP”: size $= n$ [dBdGO’97]
Background:

Introduced by the computer graphics community

“Painter’s Algorithm”  [FuchsKedemNaylor’80]

Other applications: ray shooting, solid modeling, rectangle tiling

Theoretical study initiated by Paterson and Yao, 1989
Prior Results:

$O(n \log n)$ size for $n$ segments in $\mathbb{R}^2$ [PY]

(algorithm to compute in $O(n \log n)$ time)

Conjecture: $O(n)$ size
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New lower bound of $\Omega(n \frac{\log n}{\log \log n})$ for autopartitions [Toth’00]
Prior Results:

$O(n \log n)$ size for $n$ segments in $\mathbb{R}^2$ [PY]
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Conjecture: $O(n)$ size

New lower bound of $\Omega(n \frac{\log n}{\log \log n})$ for autopartitions [Toth’00]

$O(n^2)$, $\Omega(n^2)$ for $n$ triangles in $\mathbb{R}^3$ [PY]

$O(n^{d-1})$ for $(d - 1)$-simplices in $\mathbb{R}^d$ [PY]
Prior Results (cont):

$O(n)$ size for special cases: [dBdGO’97,dBdG’94,dB’00]

- orthogonal segments in $\mathbb{R}^2$ [PY]
- fat objects
- uncluttered scenes in $\mathbb{R}^d$
- bounded length ratio segments in $\mathbb{R}^2$
- homothetic objects
- segments of $O(1)$ orientations [Toth’01]
Size of BSPs of Orthogonal (Axis-Parallel) Objects:

$\leq 3n$ for segments in $\mathbb{R}^2$  \hspace{1cm} [PY]

Lower bound: $\geq 1.25n$

$\leq 4n$ for rectangles in $\mathbb{R}^2$  \hspace{1cm} [dAF]

Implicit in d’Amore-Franciosa: $\leq 2n$ for segments

$\leq 3n$ ( $\geq 2n$ ) for rectangles in $\mathbb{R}^2$  \hspace{1cm} [BDM]

$\leq 2n$ ( $\geq 1.5n$ ) for rectangle tiling in $\mathbb{R}^2$  \hspace{1cm} [BDM]

$\Theta(n^{d/(d-1)})$ for segments in $\mathbb{R}^d$  \hspace{1cm} [PY]

$\Theta(n^{3/2})$ for rectangles in $\mathbb{R}^3$  \hspace{1cm} [PY]
Our Results:

Orthogonal (axis-parallel) objects

Tight bounds for segments in $\mathbb{R}^2$:

- Upper bound $2n - 1$
- Lower bound $2n - o(n)$

Improved lower bound of $\frac{7}{3}n - o(n)$ for rectangles in $\mathbb{R}^2$

versus $2n$ of [BDM]

New, simpler proofs (and lower constants) for upper bounds in $\mathbb{R}^3$:

- $O(n^{3/2})$ for segments
- $O(n^{3/2})$ for rectangles

New, simpler proof of $O(n^{d/(d-1)})$ for segments in $\mathbb{R}^d$
First results for \(k\)-rectangles in \(\mathbb{R}^d, \ d \geq 4\):

- If \(k < d/2\), \(\exists\) BSP of size \(O(n^{d/(d-k)})\)
  
  tight in the worst case
  
  This bound subsumes the bound \(\Theta(n^{d/(d-1)})\) for segments

- In fact, upper/lower bounds hold for \(k \leq d - 1\) for
  possibly intersecting rectangles

- Size \(O(n^{5/3}), \ \Omega(n^{5/3})\) for 2-rectangles in \(\mathbb{R}^4\)
### Summary of Results:

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($\dagger$): known bound

($\ast$): known bound, rederived here with simpler proof
BSP for Segments in $\mathbb{R}^2$: 
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[Diagram showing a binary space partitioning tree for segments in a 2D plane]
BSP for Segments in $\mathbb{R}^2$: 
Lower Bound for Segments in $\mathbb{R}^2$:

A cycle configuration of thickness $w = 5$. 
Lower Bound for Segments in $\mathbb{R}^2$: 

A $4 \times 4$ 2-grid (double grid).
Lower Bound for Segments in $\mathbb{R}^2$:

Charging scheme in a 5-grid.
Lower Bound for Rectangles (Squares) in $\mathbb{R}^2$: 
Simple Upper Bound for Segments in $\mathbb{R}^3$:

$n$ segments parallel to $x$-, $y$-, $z$-axes

For simplicity: general position (no shared coordinates)
Make \( \sqrt{n} \) slices \( \perp \) to \( z \)-axis

\( \leq \sqrt{n} \) red or green segments per slab

Each black (\( z \)-parallel) segment is cut \( \sqrt{n} \) times
Project each slab onto the \((x, y)\)-plane:

\[ \leq \sqrt{n} \text{ horizontal/vertical segments} \]

\[ \leq n \text{ points} \]
Project each slab onto the \((x, y)\)-plane:

\begin{itemize}
\item \(\leq \sqrt{n}\) horizontal/vertical segments
\item \(\leq n\) points
\end{itemize}

Cut the segments into \(O(n)\) disjoint segments (naive grid)
Project each slab onto the \((x, y)\)-plane:

\[ \leq \sqrt{n} \text{ horizontal/vertical segments} \]
\[ \leq n \text{ points} \]

Cut the segments into \(O(n)\) disjoint segments (naive grid)

Apply 2D BSP: size \(O(n)\) per slice

\(\sqrt{n}\) slices \(\to\) size \(O(n^{3/2})\) total

(In fact, each segment cut into only \(\sqrt{n}\) pieces)
Segments $E = X \cup Y \cup Z$ parallel to $x-, y-, z$-axes $x = |X|, y = |Y|, z = |Z|$, so that $x + y + z = n$.

**Theorem:** BSP size $\leq 4\sqrt[3]{xyz} + 2n - z$, for $z \leq x, y$.

$$\leq \frac{4}{3\sqrt[3]{3}} n^{3/2} + \frac{5}{3} n$$

**Lower bound:**

$$\geq \frac{1}{3\sqrt[3]{3}} n^{3/2} + n$$

**OPEN:** Close the gap in the constants
Segments in $\mathbb{R}^d$: Slice and Dice:

Make $n^{1/(d-1)}$ slices $\perp$ to $x_d$-axis

$\leq n^{(d-2)/(d-1)}$ nonvertical segments per slab

Each vertical ($x_d$-parallel) segment is cut $n^{1/(d-1)}$ times

Project each slab onto the $(x_1, \ldots, x_{d-1})$-hyperplane

- $\leq n^{(d-2)/(d-1)}$ segments in $\mathbb{R}^{d-1}$
- $\leq n$ points

Apply induction to compute BSP in each projected slab:

$$\left[n^{(d-2)/(d-1)}\right]^{(d-1)/(d-2)} = O(n)$$

$n^{1/(d-1)}$ slices $\rightarrow$ size $O(n^{d/(d-1)})$ total

(In fact, each segment cut into only $n^{1/(d-1)}$ pieces)
**k-Rectangles in \( \mathbb{R}^d \):**

\( \mathcal{R} = \) set of \( n \) axis-parallel \( k \)-dimensional hyperrectangles

Assume \( k < d/2 \), **general position**

(Thus, no pair of rectangles intersect)

Each rectangle \( r \in \mathcal{R} \) has \( k \) **extent coordinates** (project to an interval)

and \( d - k \) **fixed coordinates** (project to a point)

Rectangle \( r \) is \( x_i \)-**pass-through** in box \( K \) if \( r, K \) have same \( x_i \) projection

\( \text{pt}(r, K) = \) tuple of coordinates for which \( r \) is a pass-through in \( K \)

**Theorem** \( \text{BSP of size } O(n^{d/(d-k)}) \)
Proof – Slice and Dice:

Let \( t = cn^{1/(d-k)} \)

BSP construction in \( d \) phases:

- \( j \)-th phase: slice each cell by \( t \) hyperplanes \( \perp x_j \)-axis

Phase 1: \( t \) slices \( \perp \) to \( x_1 \)

  Result: \( t \) slabs, each with \( \leq n \) 1-PT; \( \leq n/t \) non-PT

Phase 2: \( t \) slices \( \perp \) to \( x_2 \)

  Result: \( t^2 \) subslabs, each with \( \leq n \) (1,2)-PT; \( \leq n/t \) 1-PT;

  \( \leq n/t \) 2-PT; \( \leq n/t^2 \) non-PT
**Phase \( j \):**

\[ \sigma = \text{a cell (subslab) produced by previous phases} \]

**Induction Hypothesis:** \( \forall M \subseteq \{1, \ldots, j - 1\} \text{ with } |M| \leq k, \sigma \text{ has } \leq n/t^{j-1-|M|} \text{ rectangles PT in exactly coordinates } M \)

Cut \( \sigma \) with \( O(t) \) cuts \( \perp x_j \)-axis

Ensure that \( \forall M \text{ subslab } \sigma' \) has \( \leq n/t^{j-1-|M|} \) rectangles that were in \( \sigma \) PT in exactly coordinates \( M \) and are not \( x_j \)-PT in \( \sigma' \)

(By IH, \( \sigma \) has \( \leq n/t^{j-1-|M|} \) such rectangles, so we can cut down by factor \( t \) for each \( \sigma' \))

Also, \( \forall M \subseteq \{1, \ldots, j\} \text{ with } |M| \leq k, j \in M, \sigma' \) has \( \leq n/t^{j-1-|M|} \) rectangles PT in exactly coordinates \( M \)

**Claim:** None of the final cells contains a rectangle in its interior

**Number of cells:** \( O(t^d) = O(n^{d/(d-k)}) \)
Theorem   For any $n$, $d$ and $k < d/2$, there are instances for which any
BSP has size $\Omega(n^{d/(d-k)})$
2-Rectangles in $\mathbb{R}^4$:

Apply round-robin slicing, as before, with $t = cn^{1/6}$

Result: $O(n^{2/3})$ subcells $\sigma$ with

- $\leq n/t^2 = O(n^{2/3})$ rectangles PT in 2 coord
- $\leq n/t^3 = O(n^{1/2})$ rectangles PT in exactly 1 coord
- $\leq n/t^4 = O(n^{1/3})$ rectangles not PT

**Lemma** $\sigma$ cannot contain rectangles $r, r'$ such that $r$ is PT in 2 coords and $r'$ is PT in 2 complementary coords.

Thus, only two possible values for the set

$$pt(\sigma) \equiv \{ pt(r) \mid r \text{ is a rectangle that is PT in } \sigma \text{ in 2 coords} \},$$

up to a permutation of the coordinates; namely:

(i) $pt(\sigma) = \{(1, 2), (1, 3), (2, 3)\}$

(ii) $pt(\sigma) = \{(1, 4), (2, 4), (3, 4)\}.$
Case (i): $\text{pt}(\sigma) = \{(1, 2), (1, 3), (2, 3)\}$:

All rectangles PT in $\sigma$ in 2 coords are $\perp x_4$-axis, at different heights
Cut $\sigma$ by $O(n^{1/6})$ cuts $\perp x_4$-axis so that
each subcell has $\leq n^{1/2}$ (portions of) rectangles
Continue BSP recursively in each subcell.
Let $F(n) = \text{max size of BSP the algorithm constructs}$
Overall: $O(n^{5/6})F(n^{1/2})$ cells
Case (ii): \( \text{pt}(\sigma) = \{(1, 4), (2, 4), (3, 4)\} : \\
Round-robin: \( t = O(n^{1/6}) \) cuts \( \perp x_1, x_2, \) and \( x_3 \)-axes \nEach of the \( O(n^{2/3}) \) cells cut into \( O(n^{1/2}) \) subcells \nTotal: \( O(n^{7/6}) \) subcells \nDo this so that each subcell \( \sigma \) contains at most 
\begin{itemize} 
  \item \( n^{2/3}/t^2 = n^{1/3} \) rect PT in 2 coords (one being \( x_4 \)) 
  \item \( n^{1/2}/t = n^{1/3} \) rect PT in 2 coords \textit{none} being \( x_4 \) 
  \item \( n^{1/2}/t^2 = n^{1/6} \) rect PT in exactly one coord 
  \item and \textbf{no} other rectangles. 
\end{itemize}
Lemma → the existence of 2-coord PTs of second type annihilates those PTs of first type with complementary extent coords

Thus, the extent coords of 2-coord PTs in $\sigma$ again fall into case (i) or case (ii) (with a possible permutation of the coordinates).

Case (i): proceed in a manner similar to above:
- cut $\sigma$ by $O(n^{1/6})$ cuts $\perp x_4$-axis, to get $O(n^{1/6})$ subcells, each with $\leq n^{1/6}$ rect.

Overall recursive bound:

$$O(n^{4/3}) \cdot F(n^{1/6})$$
Case (ii): with, say, \( \text{pt}(\sigma) = \{(1, 4), (2, 4), (3, 4)\} \)

Again proceed as above:

- cut \( \sigma \) in round-robin by \( t = O(n^{1/6}) \) cuts \( \perp x_1, x_2, \text{and } x_3 \)-axes

Do this to eliminate

- the \( \leq n^{1/6} \) rect that are PT in 1 coord, \textit{and}
- the \( \leq n^{1/3} \) 2-coord PTs

Hence, get BSP of size \( O(n^{1/2}) \)

Total: \( O(n^{7/6} \cdot n^{1/2}) = O(n^{5/3}) \)
Putting everything together:

\[ F(n) = O(n^{5/3}) + O(n^{4/3}) \cdot F(n^{1/6}) + O(n^{5/6}) \cdot F(n^{1/2}) \]

Solution to recurrence

\[ F(n) = O(n^{5/3} \log^c n) \]

for some appropriate constant \( c > 0 \)
Lower Bound:

**Theorem**  There exist sets of 2-rectangles in $\mathbb{R}^4$ for which any BSP has size $\Omega(n^{5/3})$
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