1 Overview

In the last lecture we learned about Binary Space Partition trees in 2 and 3 Dimension, and learned about the recent research as well as open problems in that area.

In this lecture the issue of Power Assignment in Radio Networks was discussed. The problem discussed asks us to assign transmission ranges to a given set of radios, such that any pair of radios is able to communicate and the total power consumed by all radios is minimized. We learned an efficient algorithm that solves this problem under realistic assumptions.

2 Problem Formalization

We first see the generalized version of the problem, then the problem as it occurs in real life, and finally the approximation algorithm that was presented to solve this realistic version of the problem.

2.1 Generalized Problem

The generalized problem definition is:

**Problem Definition:** Given a set of radios, each of which can be assigned an arbitrary transmission range, the set of radios that it can transmit messages to, for a given transmission range, and the cost of assigning this range to this radio, what values of transmission ranges should be assigned to each radio so that any pair of radios is able to exchange messages with possible use of intermediate nodes and the total cost incurred in assigning these transmission ranges is minimized. Mathematically, Given:

- A set of radio transmitters $R_1, R_2, ..., R_n$
- Possible transmission ranges for each transmitter $R_i : r^1_i, r^2_i, ..., r^{(n-1)}_i$ $\forall 1 \leq i \leq n$
• For each pair \((R_i, r_i^j)\) the set of radios Reachable that are within this transmission range
• the cost of assigning this transmission range to \(R_i\) : \(C(R_i, r_i^j)\).

We need to find:

• \((r_i^j \forall R_i \mid \exists \text{ a path between } R_i \text{ and } R_k \text{ in the corresponding Communication Graph}).\)

2.2 Problem solved

The problem defined above is quite general, In real world

• The cost function for all transmitters is same and is usually given by \(\text{Cost}(\text{range}) = \text{Constant} \times \text{range}^k, k \geq 2\)
• There are only two possible transmission ranges for each radio \(r_1(\text{small}), r_2(\text{large})\)
• Radios are lying in euclidean plane.
• A radio with transmission range \(r\) can transmit messages to all radios within a euclidean distance \(r\) of this node.

We assume that the Cost of assigning a range \(r = Cr^2\).

This is the exact problem that the algorithm solves. This is popularly known as the Power Assignment problem in Radio Networks.

2.3 Some General Comments

• The original problem is \textbf{NP Hard} and so is the problem solved by the proposed algorithm.
• Planar Cubic Vertex cover problem can be used to prove that the solved variant of the original problem is NP hard.
• The closest approximation bound on the number of radios assigned long range known till now on the solution was 2. The method is:
Find all connected components in the communication graph (Section 3) assuming only edges of weight $r_1$ are present.

Merge each of these connected components into a node.

Find Minimum spanning tree on the nodes formed above (put an edge between two nodes if at least one pair of radios inside them can communicate using long range.

Assign long range to one radio (that can reach some radio in the other endnode) in both endnodes of each edge in Minimum Spanning tree.

**Proof**

Clearly Optimal power assignment will assign long range to at least one radio in each node.

- The 2 approximation bound is tight. Examples can be constructed where a ratio of $2 - \epsilon$ is attained. See Figure 1 (with a sufficient number of nodes it can come arbitrarily close to 2).

- This approximation is valid even if the cost function becomes $Cr^n$ in general, this is because in terms of MST the problem is still the same, argument above will hold in that case also.

![Figure 1: In Order: Assignment by MST heuristic, Original Graph, Optimal Assignment (arrow from node $a$ to node $b$ ⇒ assign long range to one radio in $a$ that can transmit to some radio in $b$)](image)
3 Definition

3.1 Communication Graph

From a given set of radios with their locations in the euclidean plane, a communication graph is constructed as follows:

- Each Radio $R_i$ is represented by a node $n_i$ in the graph.
- There is a directed edge of weight $\min\{r_i \mid (r_i \geq \text{distance}(n_i, n_j))\}$ from $n_i$ to $n_j$

Basically from each radio we draw a circle, of radius = range, with center at radio and we draw edges to all radios lying inside that circle, and assign this edge a cost = range).

For e.g see Figure 2.

4 The Algorithm

The algorithm that we studied has an approximation ratio of $\frac{11}{6}$ with respect to the number of radios assigned longer range ($r_2$) in the optimal solution. i.e if the optimal solution OPT assigned range $r_2$ to $m$ radios than the algorithm given assigns range $r_2$ to no more than $\frac{11}{6}m$ radios.
Before looking at the algorithm let us simplify the expression of cost of the solution.
In the final solution some of the nodes (say $n_2$ nodes) shall be assigned a range $r_2$ and remaining $r_1$, so the total cost shall be,

$$\text{cost} = C(n - n_2) \cdot r_1^2 + C(n_2) \cdot r_2^2$$

$$= A(n + n_2(d^2 - 1))$$

where $n$ = total number of sensors 

$d = r_2/r_1$

$A$ is a constant

$n_2$ = number of sensors assigned long range

Now we informally describe the algorithm, A formal description of the algorithm can be found in the Paper on the same by Carmi and Katz

### 4.1 Informal Description

In this section we shall look at a detailed informal description of the algorithm and look at its execution in an example.

#### 4.1.1 Preprocessing

As a preprocessing step the algorithm first compacts all the radios that can communicate by using small range ($r_1$) only. To do so, it

1. Finds all the maximal connected components (Using Strongly Connected Component search or otherwise) in the corresponding Communication graph (consider only edges with weight $r_1$)
2. Makes a new graph ($CG$) having a node for each distinct connected component found.
3. There is an edge between two nodes $n_1$, $n_2$ in $CG$ if there exists an edge from some $n_i \in n_1$ to $n_j \in n_2$( note that the edge shall have a weight $r_2$ since graph is undirected).

For example the original graph before and after preprocessing is shown in figure 3.

After this initial processing, the algorithm works in the following two phases.

1. The graph has a cycle in it.
2. No more cycles in the graph i.e it is a tree.
4.1.2 Phase 1

We look at phase 1 of the algorithm first. In this case
Let \( C = c_1, c_2, c_l, c_1 \) be the cycle.

Note that each of \( c_i \) is a collection of radios that can communicate using the range \( r_1 \) only.
In this phase algorithm

1. Assign long range to a radio in \( c_i \) such that it can transmit to \( c_{i+1} \)
2. In \( c_l \) assign long range to a radio that can transmit to some radio in \( c_1 \).
3. After the above assignments are done, we compress the cycle into a single node in the graph.
4. Repeat the above process till there are no more cycles in the graph,

Now we have a tree (since otherwise it means that the graph is not connected \( \Rightarrow \) no solution is possible)

Figure 4 shows execution of phase 1 on the graph.

4.1.3 Phase 2

In Phase 2 we use a \( \frac{4}{3} \) approximation (It assigns long range to no more radios than \( \frac{4}{3} m \),
where \( m \) denotes the number of radios assigned long range in optimal solution to this tree).

Let there be \( k \) nodes in the tree formed after Phase 1.
The optimal solution must assign long range to at least $k$ and at most $2(k-1)$ radios. (see Section 2.3).

Let $P$ be a node (component, collection of radios strongly connected by range $r_1$) in the graph, then the maximum number of nodes assigned range $r_2$ by $P$ is 5 (this is a property of disk packing in euclidean plane).

It is known that if the size of subsets is $\leq k$, then the set cover problem can be solved in polynomial time to within a $\sum_{i=1}^{k} (1/i) - 1/2$ approximation. For example, if $k \leq 5$ it becomes: 107/60.

The set cover problem can be solved optimally in polynomial time if all the subsets are of size 2 or less.

Now let us look at the way trees are handled. We assume that the tree has $k$ nodes and the optimal assignment requires long range for $m$ radios. For each component (node) $C$ in the graph, and any radio $P$ in it let

$\chi_c$ denote the number of children of $C$

$m_c$ denote the number of long range assignments made in the optimal solution in this component.

$d_p$ denote number of yet unsatisfied children of $C$ that will be satisfied if $P$ is assigned long range.

We note that

$\sum_c \chi_c = k - 1$ and $\sum_c m_c = m$
Now the procedure we follow shall assign at most $\frac{1}{3} \chi_c + m_c$ radios a long range ($r_2$) in $C$. The assignments are made as follows:

**Case 1:** $\chi_c \leq 2$

Solve $C$ optimally and use $m_c$ long range assignments (consider all cases and find the optimal assignment)

*Cost incurred is $m_c$ only.*

**Case 2:** $\chi_c \geq 3$

1. If $\exists$ a radio in $C$ that covers 3 (or more) uncovered children then assign long range to this radio.
2. Repeat this till no radio covers 3 (or more) uncovered children.
3. Let $X$ be the number of uncovered children now.
4. Now all radios either cover 2 children or 1 child. This problem can be solved optimally (the set to cover is the set of all children, and the subsets are the group of children covered by each radio).
5. If the parent is not already covered, cover it by choosing a radio that can reach it.

*Steps 1 and 2 lead to a maximum increase in incurred cost of $\frac{1}{3}(\chi_c - X)$.*

*Step 4 adds a cost of at most $m_c$ only.*

*Step 5 adds a cost of at most 1*

Now if $X \leq 2$ then we solve optimally, in this case cost incurred is $\frac{1}{3}(\chi_c - X) + m_c \leq \frac{1}{3} \chi_c + m_c$

and if $X \geq 3$ then $\frac{1}{3}(\chi_c - X) + 1 + m_c \leq \frac{1}{3} \chi_c + m_c$.

Figure 5 shows the execution of entire algorithm on the example graph.

5 Approximation Ratio

5.1 $\frac{4}{3}$ when Graph is a tree

**Claim** If the graph is a tree then the above algorithm (Section 4.1.3) has an approximation ratio of $\frac{4}{3}$ for number of radios assigned long range.

**Proof:** At any node $C$ we assign $\leq \frac{1}{3} \chi_c + m_c$ radios long range. So, total number of radios assigned long range
shall be \( \leq \sum_c (\frac{1}{3}x_c + m_c) \leq \frac{4}{3}m \)

since total number of children in tree is only \( k - 1 \).

5.2 \( \frac{11}{6} \) approximation ratio for the complete Algorithm

Claim: The above algorithm (Section 4) has an approximation ratio of 11/6 for number of radios assigned long range.

Proof: Let \( i \) be the number of cycles found in Phase 1 of the algorithm.
We assume that all cycles were of length 3 (if they had a longer length, then it only helps us, look at the proof below)
At end of Phase 1 we shall have \( (k - 2i) \) nodes in the graph (if cycles had longer length, then this number will be even smaller)
(remember \( k \) = no of nodes in compressed graph at end of Phase 1)
Now consider the following two (exhaustive) cases:

Case 1: \( i > \frac{k}{2} - \frac{m}{3} \)
In this case, during Phase 2, we assign \( 2(k - 2i - 1) \) radios a long range (we can use the 4/3 approximation also, but proving the bound does not require that) This obviously covers the entire graph. Therefore total number of nodes with long range
\[ 3i + 2(k - 2i - 1) \leq 2k - i \leq 2k - \left( \frac{k}{2} - \frac{m}{3} \right) \leq \frac{3k}{2} - \frac{m}{3} \leq \frac{11}{6}m \] (remember \( k \leq m \))

**Case 2:** \( i \leq \frac{k}{2} - \frac{m}{3} \)

In this case during Phase 2 assign long range using the \( \frac{4}{3} \) approximation:

Therefore total number of nodes with long range \( \leq 3i + 4m/3 \leq \frac{11}{6}m \)

Hence the bound is proved. We note that it is still an open question as to whether the above bound is tight or not.