1 Overview

In this lecture, we introduce the concept of Davenport-Schinzel sequences. We start by defining them and giving some results for the bounds on the length of these sequences, followed by exploring a few of their geometric applications.

2 Definition

Let $n, s$ be positive integers. A sequence $U = \langle u_1, \cdots, u_m \rangle$ of symbols (derived from an alphabet $\Lambda$ of size $n$) is an $(n, s)$ Davenport-Schinzel sequence (a $DS(n, s)$-sequence for short), if it satisfies the following conditions:

1. For each $i < m$, we have $u_i \neq u_{i+1}$.

2. There do not exist $s + 2$ indices $1 \leq i_1 < i_2 < \cdots < i_{s+2} \leq m$ such that $u_{i_1} = u_{i_3} = u_{i_5} = \cdots = a, u_{i_2} = u_{i_4} = u_{i_6} = \cdots = b$, and $a \neq b$. In other words, there is no sub-sequence of two alternating symbols of length greater than or equal to $s + 2$. We refer to $s$ as the order of the sequence.

2.1 Example

$\langle 1, 2, 1, 3, 2, 1, 2, 3, 1, 4, 2, 3, 2, 1 \rangle$ would be a $DS(4, 8)$ sequence. It doesn’t have any two consecutive symbols which are same and it has no sub-sequence of two alternating symbols of length greater than or equal to 10.
3 Maximum Length of a DS(n,s) Sequence

We will write $|U| = m$ for the length of the sequence $U$. Define

$$\lambda_s(n) = \max \{ m | U \text{ is a DS(n,s)-sequence} \}.$$ 

Following are some results for bounds on $\lambda_s(n)$ for varying $s$:

3.1 DS(n,1)-Sequences

In such sequences, each of the symbols can appear atmost once. If any symbols is repeated at index $i$ and index $j$, then there should be another symbol between $u_i$ and $u_j$ (due to property 1). Let $i < k < j$. Then, since $u_i = u_j$, $< u_i \ldots u_k \ldots u_j >$ form an alternating sequence of length 3, preventing $U$ from being an order 1 sequence. So, the maximum length of an order 1 DS-sequence is $n$.

$$\lambda_1(n) = n.$$

3.2 DS(n,2)-Sequences

In this subsection, we prove that $\lambda_2(n) = 2n - 1$. We first start with giving a proof by induction for the claim that $\lambda_2(n) \leq 2n - 1$. We then follow it up with a construction, which establishes that $\lambda_2(n) \geq 2n - 1$, thereby proving that $\lambda_2(n) = 2n - 1$.

3.2.1 $\lambda_2(n) \leq 2n - 1$

- Base Case
  $$\lambda_2(1) = 1.$$ So, the statement is true for $n = 1$.

- Induction Hypothesis
  Assume the claim to be true for $n = k - 1$. So, $\lambda_2(k - 1) \leq 2k - 3$.

- Induction Step
  Prove the claim for $n = k$:
  Let $U$ be a DS(2,k)-sequence of maximum possible length. So, $|U| = \lambda_2(k)$. Let $\mu_a = \text{First index in } U = < u_1, u_2, \ldots, u_m > \text{ where } a \text{ occurs } (u_{\mu_a} = a)$. Let $b$ be the symbol which maximises $\mu_b$, i.e. $\mu_b \geq \mu_a \forall a \in \Lambda$ ($\Lambda$ is the alphabet from which the symbols
Lemma: $b$ occurs only once in $U$.

Proof: Proof by contradiction. Let $b$ were to appear twice in $U$. The two occurrences of $b$ can’t be consecutive, so let there be another symbol $c$ between the two occurrences of symbol $b$. Now, the first occurrence of $c$ should be before that of $b$, as $\mu_b > \mu_c$ from the way we have chosen $b$. This gives us an alternating sub-sequence of length $4 < c \ldots b \ldots c \ldots b$, preventing $U$ from being an order 2 DS sequence. Thus, if $\mu_b \geq \mu_a \forall a \in \Lambda$, then $b$ can appear only once in $U$.

Now, if we delete $b$ from $U$, and $U$ looks like $\ldots xbx \ldots$, then we have to remove one of the occurrences of $x$ for the remaining list to be still a DS sequence. If $b$ is not flanked on both sides by the same symbol, then we can safely remove it and the remaining sequence, which would be a sequence on $k - 1$ symbols, would still be a DS sequence. So

$$\lambda_2(k) \leq \lambda_2(k - 1) + 2 \leq (2k - 3) + 2 = 2k - 1.$$  

**Hence,** $\lambda_2(n) \leq 2n - 1$.

### 3.2.2 $\lambda_2(n) \geq 2n - 1$

Given $n$ symbols, $a_1, a_2, \ldots, a_n$, we can construct a DS(n,2)-sequence of length $2n - 1$. Consider the sequence $U = < a_1, a_2, \ldots, a_n, a_{n-1}, a_{n-2}, \ldots, a_1 >$. Now, $|U| = 2n - 1$, and $U$ satisfies the two properties for being a DS(n,2)-sequence. So, we have constructed a DS(n,2)-sequence of length $2n - 1$. Thus, $\lambda_2(n) \geq 2n - 1$.

**The above two subsections imply that** $\lambda_2(n) = 2n - 1$.

### 3.3 DS(n,3)-Sequences

Davenport-Schinzel showed in 1965 that $\lambda_3(n) \leq 2n(ln(n) + O(1))$.

**Proof:** Let $U$ be the longest possible DS(n,3)-sequence. So, $|U| = \lambda_3(n)$. So, on an average, each symbol occurs $\frac{|U|}{n} \leq \frac{\lambda_3(n)}{n}$ times. Let $a$ be the symbol with fewest occurrences in the given $U$. 

3
\[ \therefore \text{number of occurrences of } a \leq \frac{\lambda_3(n)}{n}. \]

Now, remove \( a \) from \( U \). This step will induce some adjacent pairs of same symbols in the remainder of the list. For example, if \( U = \langle 1, 2, 3, 2, 4, 5, 3, 5, 6, 3, 6 \rangle \), then removal of all the occurrences of 3 from the list will give an adjacent pair each of 2, 6 and 5 in the remaining list.

Lemma: If all the occurrences of a symbol \( a \) are removed from a DS\((n,3)\)-sequence, the remaining list consists of at most two adjacent pairs of same symbols, namely those induced by the removal of first and last occurrences of \( a \) from the original list.

Proof: Proof by contradiction. Assume there is an adjacent pair of same symbols in the remainder of the list, which is not induced by either the first, or the last occurrence of \( a \). So then, the original list \( U \) would look like \( \langle \ldots a \ldots xax \ldots a \ldots \rangle \). This sequence has an alternating sub-sequence of length 5, contradicting the fact that \( U \) is an order-3 DS sequence.

Thus, removing all the occurrences of \( a \) and at most 2 other symbols from \( U \) gives a valid DS\((n-1,3)\)-sequence. So

\[
\lambda_3(n) \leq \lambda_3(n - 1) + \frac{\lambda_3(n)}{n} + 2
\]

\[\Rightarrow \lambda_3(n)(1 - \frac{1}{n}) \leq \lambda_3(n - 1) + 2 \]

\[\Rightarrow \frac{\lambda_3(n)}{n} \leq \frac{\lambda_3(n - 1)}{n - 1} + \frac{2}{n - 1} \]

\[\Rightarrow \frac{\lambda_3(n - 1)}{n - 1} \leq \frac{\lambda_3(n - 2)}{n - 2} + \frac{2}{n - 2} + \frac{2}{n - 1} \]

\[\vdots \]

\[\Rightarrow \frac{\lambda_3(n)}{n} \leq \frac{\lambda_3(1)}{1} + \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \ldots + \frac{2}{n - 1} \]

\[\Rightarrow \frac{\lambda_3(n)}{n} \leq O(1) + 2ln(n) \]

\[\lambda_3(n) \leq 2n(ln(n) + O(1)) \]

- Note: The tight bound for \( \lambda_3(n) \) has been achieved since, and is as follows:
  \[\lambda_3(n) = \Theta(n\alpha(n))\]
3.4 Higher Order DS-Sequences

Following table gives the bounds for the maximum lengths of higher order DS-sequences:

<table>
<thead>
<tr>
<th>Order of the sequence(s)</th>
<th>Bound on Length((\lambda_s(n)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(\Theta(n \alpha(n)))</td>
</tr>
<tr>
<td>4</td>
<td>(\Theta(n^2 \alpha(n)))</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(2s)</td>
<td>(O(n^2 \Omega(\alpha(n)^{s-1})))</td>
</tr>
<tr>
<td>(2s + 1)</td>
<td>(\Omega(n^2 \Omega(\alpha(n)^{s-1})))</td>
</tr>
</tbody>
</table>

4 Motivation: Lower Envelopes

Let \(F = \{f_1, f_2, \ldots, f_n\}\) be a collection of \(n\) real-valued continuous functions defined on a common interval \(I\). Suppose that for each \(i \neq j\), the functions \(f_i\) and \(f_j\) intersect in at most \(s\) points (as an example, this would be the case with polynomials of degree \(s\)). The lower envelope of \(F\) is defined as:

\[
L(x) = \min(f_i(x)), 1 \leq i \leq n, x \in I
\]

Let \(m\) be the smallest number of subintervals \(I_1, I_2, \ldots, I_m\) of \(I\) such that for each \(I_k\), there exists an index \(u_k\) with \(L(x) = f_{u_k}(x)\) for all \(x \in I_k\). In other words, \(m\) is the number of (maximal) connected portions of the graphs of the \(f_i\)’s which constitute the graph of \(L(x)\). The endpoints of the intervals \(I_k\) are called the breakpoints or the transition points of the envelope \(L(x)\). Assuming that \(I_1, I_2, \ldots, I_m\) are arranged in this order from left to right, put

\[
U(f_1, f_2, \ldots, f_n) = < u_1, \ldots, u_m >
\]

See the following figure for an illustration of this "lower envelope sequence."
4.1 Lower Envelopes and DS Sequences

**Lemma** $U(f_1, \ldots, f_n)$ is a $DS(n, s)$-sequence.

**Proof:** For the first property, note that, by definition, $U = U(f_1, \ldots, f_n)$ does not contain a pair of adjacent identical elements (as the sub-intervals are taken to be maximal connected portions).

For second property, suppose that there exist two distinct indices $a \neq b$ so that $U$ contains an alternating sub-sequence of $a$ and $b$ of length $s + 2$. By definition of the lower envelop, this would imply existence of $s + 2$ sub-intervals of $I$ ($I_1, I_2, \ldots, I_{s+2}$), such that $f_a(x) < f_b(x)$ for $x \epsilon (I_1 \cup I_3 \cup \ldots)$ and $f_b(x) < f_a(x)$ for $x \epsilon (I_2 \cup I_4 \cup \ldots)$. Since $f_a$ and $f_b$ are continuous and their lower envelop forms $s + 2$ distinct maximal connected intervals, there must exist $s + 1$ distinct points where $f_a$ crosses $f_b$. This contradicts the fact that $f_a$ and $f_b$ intersect in at most $s$ points. So

\[ U(f_1, \ldots, f_n) \text{ is a } DS(n, s)\text{-sequence.} \]
5 Geometric Applications for DS Sequences

DS sequences, as illustrated above, find their applications in finding the complexity of lower envelopes of a collection of curves. Following are some of the examples:

- **Envelop of n lines**: Lines are degree \(1\) curves, so the lower envelop of \(n\) lines would consist of at most \(\lambda_1(n) = n\) distinct maximal intervals, such that each of these intervals is a part of a single line. Hence the complexity of the lower envelop of \(n\) lines is \(n\).
  This bound may also be applied to find the complexity of a face in an arrangement of \(n\) lines.

- **Lower Envelop of n parabolas**: Consider \(n\) parabolas, all of them opening towards the positive \(y\) direction. Since, a parabola is a degree-2 curve, so the lower envelop of this arrangement of \(n\) parabolas would consist of at most \(\lambda_2(n) = 2n - 1\) distinct maximal intervals, such that each of these intervals is a part of a single parabola. This kind of arrangement arises in the Fortune Sweep algorithm to find the voronoi diagram of \(n\) points. This bound helps in proving the running time of the aforesaid algorithm.

- **Lower Envelop of n curve segments**: As is illustrated in the figure, lower envelop of two line segments can have 4 separate(maximal) sub-intervals such that each belongs to one of the segments. So, 2 line segments can be said to intersect in 3 different points. So, the lower envelop of an arrangement of \(n\) line segments would consist of at most \(\lambda_3(n) = \Theta(n\alpha(n))\) distinct maximal intervals, such that each of these intervals is a part of a single line segment.
So, although the lower envelop of $n$ degree-$s$ curves will correspond to a DS($n,s$)-sequence, the lower envelop of $n$ degree-$s$ curved segments will correspond to a DS($n,s+2$)-sequence due to the induction of two extra transition points (at the ends of the curved segments).