Witness Rectangle Graphs*

Boris Aronov† Muriel Dulieu* Ferran Hurtado‡

Abstract

In a *witness rectangle graph* (WRG) on vertex point set P with respect to witness point set W in the plane, two points x, y in P are adjacent whenever the open rectangle with x and y as opposite corners contains at least one witness point in W. WRGs are representative of a much larger family of witness proximity graphs introduced in two previous papers.

We study graph-theoretic properties of WRGs. We prove that any WRG has at most two non-trivial connected components. We bound the diameter of the non-trivial connected components of a WRG in both the one-component and two-component case. In the two-component case, we prove that a graph is representable as a WRG if and only if each component is a co-interval graph, thereby providing a complete characterization of WRGs of this type. We also completely characterize trees drawable as WRGs.

Finally, we conclude with some related results on the number of points required to stab all the rectangles defined by a set of n points.

1 Introduction

Proximity graphs have been widely used in situations in which there is a need of expressing the fact that some objects in a given set—which are assigned to nodes in the graph—are close, adjacent, or neighbors, according to some geometric, physical, or conceptual criterion, which translates to edges being added to the corresponding graph. In most proximity graphs, given a point set P, the adjacency between two points $p, q \in P$ is decided by checking whether in their *region of influence* there is no other point from P, besides p and q. There are many variations, depending mainly on the choice of the family of influence regions [5,8,10].

A recently introduced generalization is the concept of *witness proximity graphs* [1, 2], in which the adjacency of points in a given vertex set P is decided by the presence or absence of points from a second point set W—the *witnesses*—in their region of influence. In the *positive witness* version, there is an adjacency when a witness point is covered by the region of influence. This generalization includes the classic proximity graphs as a particular case, and offers both a stronger tool for neighborhood description and much more flexibility for graph representation purposes.

In this paper, we consider a positive witness proximity graph related to the rectangle-of-influence graph, the witness rectangle graph. In the *rectangle of influence graph* RIG(P), usually studied as one of the basic proximity graphs [10, 11], $x, y \in P$ are adjacent when the *rectangle* $B(x, y)$ *they define* covers no third point from P; $B(x, y)$ is the unique open isothetic rectangle with x and y at its opposite corners.

The *witness rectangle graph* (WRG) of vertex point set P (or, simply, *vertices*) with respect to witness point set W (*witnesses*), denoted $\mathrm{RG}^+(P, W)$, is the graph with the vertex set P, in which two points $x, y \in P$ are adjacent when the rectangle $B(x, y)$ contains at least one point of W.

The graph $\mathrm{RG}^+(P, \varnothing)$ has no edges. When W is sufficiently large and appropriately distributed, $\mathrm{RG}^+(P, W)$ is complete.

Terminology and notation. Throughout the paper, we will work with finite point sets in the plane, in which no two points lie on the same vertical or the same horizontal line.

Hereafter, for a graph $G = (V, E)$ we write $xy \in E$ or $x \sim y$ to indicate that $x, y \in V$ are adjacent in G, and generally use standard graph terminology as in [4]. When we speak of a *non-trivial connected component* of a graph, we refer to a connected component with at least one edge (and at least two vertices).

*Research of B.A. has been partially supported by NSA MSP Grants H98230-06-1-0016 and H98230-10-1-0210. Research of B.A. and M.D. has also been supported by a grant from the U.S.-Israel Binational Science Foundation and by NSF Grant CCF-08-30691. Research by F.H. has been partially supported by projects MEC MTM2006-01267 and MTM2009-07242, and Gen. Catalunya DGR 2005SGR00692 and 2009SGR1040.

†Department of Computer Science and Engineering, Polytechnic Institute of NYU, Brooklyn, New York, USA. *aronov@poly.edu*, *mdulieu@gmail.com*

‡Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya (UPC), Barcelona, Spain. *ferran.hurtado@upc.edu*

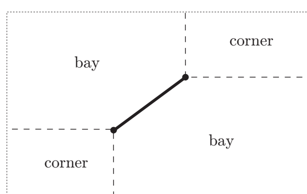
How to compute a witness rectangle graph. A simple modification of the algorithm of De Berg, Carlsson, and Overmars [3] yields:

Theorem 1 (De Berg, Carlsson, and Overmars [3]). *Let P and W be two point sets in the plane. The witness rectangle graph* $\mathrm{RG}^+(P,W)$ *with k edges can be computed in $O(k + n \log n)$ time, where $n := \max\{|P|,|W|\}$.*

2 Structure of WRGs

Let $G := \mathrm{RG}^+(P,W)$ be the witness rectangle graph of vertex set P with respect to witness set W. We assume that the set of witnesses is *minimal*, in the sense that removing any one witness from W changes G. Put $n := \max\{|P|,|W|\}$ and let $E := E(G)$ be the edge set of G. We partition E into E^+ and E^- according to the slope sign of the edges when drawn as segments. Slightly abusing the terminology we refer to two edges of E^+ (or two edges of E^-) as having *the same slope* and an edge of E^+ and an edge of E^- as having *opposite slopes*.

Recall that the open isothetic rectangle (or *box*, for short) defined by two points p and q in the plane is denoted $B(p,q)$; for an edge $e = pq$ we also write $B(e)$ instead of $B(p,q)$. Every edge e, say in E^+, defines four regions as in the figure on the right, that we call *(open) corners* and *(closed) bays*.



Note that for any P, W, and $P' \subset P$, the graph $\mathrm{RG}^+(P',W)$ is an induced subgraph of $\mathrm{RG}^+(P,W)$, so the class of graphs representable as WRGs is closed under the operation of taking induced subgraphs.

Two edges are *independent* when they share no vertices and the subgraph induced by their endpoints contains no third edge. Below we show that G cannot contain three pairwise independent edges, which imposes severe constraints on the graph structure of G.

Lemma 1. *Two independent edges in E^+ (respectively, E^-) cannot cross or share a witness. The line defined by their witnesses is of negative (respectively, positive) slope.*

Proof. Let the two edges be $ab, cd \in E^+$, with $x(a) < x(b)$ and $x(c) < x(d)$. A common witness would have a and c in its third quadrant and b and d in the first, implying $a \sim d$ and $c \sim b$, a contradiction. If ab and cd crossed, assume without loss of generality that $x(a) < x(c)$. Neither c nor d can be inside $B(a,b)$ (because a or b would be in a corner of cd, therefore the edges would not be independent) and hence $B(a,d) \cup B(c,b) \supset B(a,b)$, implying $a \sim d$ or $c \sim b$, a contradiction. Finally, the second part of the statement follows from the fact that every vertex inside a corner of an edge e is adjacent to at least one endpoint of e. □

Lemma 2. *Two independent edges with opposite slopes must share a witness.*

Proof. Let $ab \in E^+$ and $cd \in E^-$ be independent. Let w be a witness for ab and let w' be a witness for cd. The points c and d are not in quadrants I or III of w, as otherwise the two edges would not be independent. If w is shared, we are done. Otherwise it cannot be that c lies in quadrant II of w while d lies in quadrant IV, or vice versa. Therefore c and d are either both in quadrant II or both in quadrant IV of w. Assume, without loss of generality, the former is true. The witness w' is not outside of $B(a,b)$, as we would have $c \sim a$ and/or $c \sim b$ (assuming, without loss of generality, that $x(c) < x(d)$) and the edges would not be independent. Therefore w' is in $B(a,b)$, so w' is a shared witness, as claimed. □

Lemma 3. *There are no three pairwise independent edges in E^+ (or in E^-).*

Proof. Assume that three pairwise independent edges e_1, e_2, e_3 in E^+ are witnessed by w_1, w_2, w_3, respectively, with $x(w_1) < x(w_2) < x(w_3)$. Then, by Lemma 1, $y(w_1) > y(w_2) > y(w_3)$. By the same lemma at least one endpoint of e_1 is in the second quadrant of w_2 and at least one endpoint of e_3 is in its fourth quadrant, contradicting their independence. □

Theorem 2. *G does not contain three pairwise independent edges.*

Proof. By Lemma 3, two edges ab and cd of the three pairwise independent edges ab, cd, and ef have opposite slopes. By Lemma 2, ab and cd share a witness point w. Every quadrant of w contains one of the points a, b, c, or d, therefore both e and f must be adjacent to one of them, a contradiction. □

The preceding results allow a complete characterization of the trees that can be realized as WRGs. An analogous result for rectangle-of-influence graphs was given in [11].

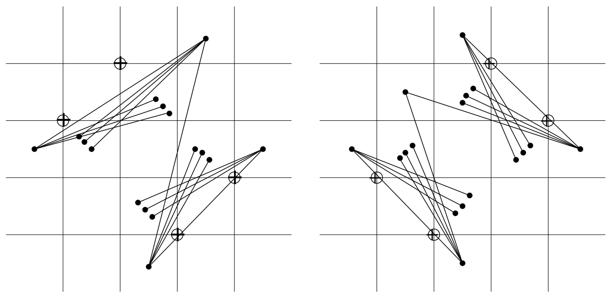


Figure 1: All WRG trees have one of these two forms. Any number of vertices can be added in the regions containing three vertices.

Theorem 3. *A tree is representable as an WRG if and only if it has no three independent edges.*

Sketch of the proof. Examine all combinatorial trees without three independent edges. This requires a case analysis that we omit in this version. Any such tree is a subtree of one of the two maximal trees depicted in Figure 1, both of which happen to be representable as WRGs, as seen in the figure. □

Lemma 2 immediately implies the following structural result that is far from complete characterization, yet narrows substantially the class of graphs representable as WRGs.

Theorem 4. *A WRG has at most two non-trivial connected components. If there are exactly two, each has diameter at most three. If there is one, its diameter is at most six.*

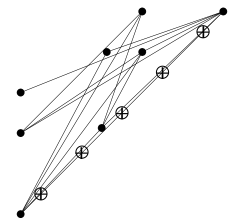
Note that the bounds on the diameter are tight: the tree in Figure 1 (right) has diameter six and it is easy to draw the disjoint union of two three-link paths as a WRG, by removing one vertex from Figure 1 (right), for example.

3 Two connected components

In this section we define a subclass of witness rectangle graphs, called *staircase graphs*. We argue that a WRG with precisely two non-trivial connected components has a very rigid structure. Namely, each of its non-trivial connected components is isomorphic to a staircase graph.

Definition. *A* staircase graph of type IV *is a witness rectangle graph, such that the witnesses form an ascending chain (i.e., for every witness, other witnesses lie only in its quadrants I and III) and all the vertices lie above the chain (i.e., quadrant IV of every witness is empty of vertices); refer the figure below. Staircase graphs of types I, II, and III are defined analogously; they are rotated versions of the above. The type of the staircase graph is determined by which quadrant of all witnesses is empty of vertices.*

Note that an induced subgraph of a staircase graph is a staircase graph (of the same type).



Theorem 5. *In a witness rectangle graph with two non-trivial connected components, each component is isomorphic to a staircase graph. Conversely, the disjoint union of two graphs representable as staircase graphs is isomorphic to a witness rectangle graph.*

Proof. Omitted in this version. □

4 What are staircase graphs, really?

The above discussion is quite unsatisfactory in that it describes one new class of graphs in terms of another such new class. In this section, we discover that the class of graphs representable as staircase graphs is really a well known family of graphs.

Recall that an *interval graph* is one that can be realized as the intersection graph of a set of intervals on a line, i.e., its set of vertices can be put in one-to-one correspondence with a set of intervals, with two vertices being adjacent if and only if the corresponding intervals intersect. A *co-interval graph* is the complement of an interval graph, i.e., a graph representable as a collection of intervals in which adjacent vertices correspond to disjoint intervals.

Theorem 6. *Graphs representable as staircase graphs are exactly the co-interval graphs.*

Proof. Consider a vertex v in a staircase graph $\mathrm{RG}^+(V, W)$. Without loss of generality, assume the witnesses W lie on the line $\ell\colon y = x$ and the vertices lie above it. Create an artificial witness on ℓ lying above all vertices. Associate v with the smallest interval I_v of ℓ containing all witnesses lying to the right and below v as well as the witness immediately above v. It is easily checked that $v \sim v'$ in $\mathrm{RG}^+(V, W)$ if and only if I_v and $I_{v'}$ do not meet. Thus the intersection graph of the intervals $\{I_v \mid v \in V\}$ is isomorphic to the complement of $\mathrm{RG}^+(V, W)$.

Conversely, let H be a co-interval graph on n vertices and let $\{I_v\}$ be its realization by a set of intervals on the line $\ell\colon x = y$. Extend each interval, if necessary, slightly, to ensure that the $2n$ endpoints of the intervals are distinct. Place $2n-1$ witnesses along ℓ, separating consecutive endpoints, and transform each interval $I_v = (a_v, a_v), (b_v, b_v)$ into point $p_v = (a_v, b_v)$. Let W and P be the set of witnesses and points thus generated. Now I_v misses I_w if and only if $a_v < b_v < a_w < b_w$ or $a_w < b_w < a_v < b_v$, which happens precisely if and only if the rectangle $B(p_v, p_w)$ contains a witness. Hence $\mathrm{RG}^+(P, W)$ is isomorphic to H, as claimed. □

Theorems 5 and 6 imply the following much more satisfactory statement:

Theorem 7. *The class of graphs representable as witness rectangle graphs with two non-trivial connected components is precisely the class of graphs formed as a disjoint union of zero or more isolated vertices and two co-interval graphs.*

Corollary 8. *Whether or not a given combinatorial graph $G = (V, E)$ with two non-trivial connected components can be drawn as a WRG can be tested in time $O(|V| + |E|)$.*

Proof. Use the linear-time recognition algorithm for co-interval graphs from [7]. □

5 How to stab rectangles

Let P be a set of n points in the plane, and let S be some given family of geometric regions, each with at least two points from P on its boundary. The problem of how many points are required to stab all the elements of S using a second set W of points has been considered several times for different families of regions [1, 2, 6, 9, 12].

We consider here a variant of this problem that is related to WRGs, in which we focus on the family R of all open isothetic rectangles containing two points of P on their boundary and assume that the points of P have no repeated x- or y-coordinates. We denote by $st_R(n)$ the maximum number of piercing points that are required, when all sets P of n points are considered. Stabbing all the rectangles that have p and q on their boundary is equivalent to just stabbing $B(p, q)$. Therefore we see that

$$st_R(n) = \max_{|P|=n} \min\{|W| : \mathrm{RG}^+(P, W) = K_{|P|}\}.$$

Theorem 9. $st_R(n) = 2n - \Theta(\sqrt{n})$.

Proof. Omitted in this version. □

References

[1] B. Aronov, M. Dulieu, and F. Hurtado, "Witness (Delaunay) graphs," submitted for publication. Extended abstract in *Proc. 7th Japan Conf. Comput. Geometry and Graphs*, pp. 87–88, 2009.

[2] B. Aronov, M. Dulieu, and F. Hurtado, "Witness Gabriel graphs," *Computational Geometry: Theory and Applications*, accepted for publication. Extended abstract in *Proc. 25th European Workshop Comput. Geometry*, pp. 13–16, 2009.

[3] M. de Berg, S. Carlsson, and M. Overmars, "A general approach to dominance in the plane," *J. Algorithms* 13(2), 274–296, 1992.

[4] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman & Hall, 2004 (4th edition).

[5] S. Collette. "Regions, Distances and Graphs". PhD thesis, Université Libre de Bruxelles, 2006.

[6] J. Czyzowicz, E. Kranakis, and J. Urrutia, "Dissections, cuts, and triangulations," *Proc. 11th Canadian Conf. Comput. Geometry*, pp. 154-157, 1999.

[7] M. Habib, R. McConnell, C. Paul, and L. Viennot, "Lex-BFS and partition refinement, with applications to transitive orientation, interval graph recognition and consecutive ones testing," *Theoretical Computer Science*, 234(1–2), 59–84, 2000.

[8] J.W. Jaromczyk and G.T. Toussaint, "Relative neighborhood graphs and their relatives," *Proc. IEEE*, 80(9), 1502–1517, 1992.

[9] M. Katchalski and A. Meir, "On empty triangles determined by points in the plane," *Acta Math. Hungar.*, 51, 23–328, 1988.

[10] G. Liotta, Proximity Drawings. Chapter in *Handbook of Graph Drawing and Visualization*, R. Tamassia, Editor, Chapman & Hall/CRC Press, in preparation.

[11] G. Liotta, A. Lubiw, H. Meijer, and S. Whitesides, "The rectangle of influence drawability problem," *Comput. Geometry Theory Appl.*, 10(1), 1–22, 1998.

[12] T. Sakai and J. Urrutia, "Covering the convex quadrilaterals of point sets," *Graphs and Combinatorics*, Vol. 23 Suppl. 1, 343–357, 2007.