Inversion of Quadratic Bézier Triangles*

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Abstract

We present a test for the invertibility of second order Bézier elements. Our test that can determine precisely if such the function is invertible in only a constant number of operations. Although our formulation is specific to triangles, it is simple to extend it to other polygonal domains. Our test is used in the routines of the Lagrangian finite element fluid solver Tumble. Tumble uses moving mesh algorithms that might cause elements to become inverted such that the fluid's velocity field is not reconstructible. Our test is used to ensure that elements in the mesh are invertible after each operation. Furthermore, we present possible invariants that might allow certain operations to be performed without the need to test for invertibility at every step. In turn, we present pathological examples of Bézier triangles that fail under naive invariants.

Introduction

Bézier elements have use in finite element method to represent a polynomial mapping from a linear unit element such as a triangle or square. A set of Bézier elements is often constructed to form a mesh. Operations on Bézier meshes include moving the control structure under invariants, or reconstructing the mesh entirely. In finite element method in particular, a common invariant is that it is necessary for the Bézier polynomials to be invertible, so as to make reconstructing data possible. Often stronger and easier to ensure invariants are used in the place of precise invertibility. This however, sacrifices the set of operations that can be performed and thus also the precision of the calculation. It is thus desirable to have precise tests for invertibility.

We present a simple test for determining the invertibility of a second degree polynomial function defined on a triangle. Our test can determine precisely if such a function is invertible in only a constant number of arithmetic operations. It is possible to show that a function is invertible iff the determinant of its Jacobian is positive everywhere on its domain. By simply performing calculus on the determinant of the Jacobian of the Bézier polynomial, we find that it is unnecessary to determine the minimum within the domain in the case of second degree Bézier triangles. We can determine whether a Bézier triangle is invertible by checking the global minimum and the minimum along the edges, the former of which is solvable by matrix inversion, and the latter of which is solvable using the quadratic equation. This test has been successfully implemented in the subroutines of the Lagrangian finite element fluid solver Tumble, and has been shown to increase the length of the simulation before stability is lost or elements become inverted.

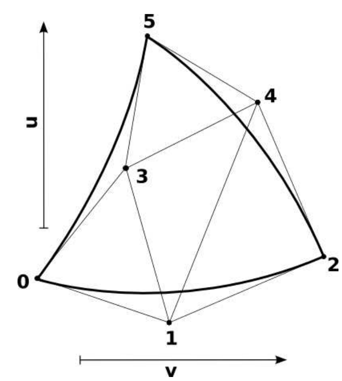

Figure 1: An example of a Bézier triangle

Given this test, special cases have been observed for the purpose of designing more powerful invariants. During a fluid simulation, vortexes might occur in the fluid, causing the surrounding mesh to curve undesirably to the point of inversion. To prevent this, we attempt to straighten the misbehaving edges. However, straightening an edge does not necessarily preserve invertibility on the triangle for which that edge is being made more concave. To rectify this, we would like to impose the condition that the convex hulls of the edge's control meshes are disjoint. While this condition is not sufficient for invertibility, it is seemingly sufficient for preserving invertibility when straitening a convex edge. To prove this as well as simpler tests for invertibility and conjectures, we

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conjecture that if $\det(J) > 0$ everywhere on the boundary of the domain triangle, the boundary of domain triangle has a winding number of 1, then if the function is second degree, it must be invertible everywhere within the domain. Because straitening edges preserves the disjoint edge hull condition, using it as an invariant, would reduce the need for the invertibility test and allow simulations to run for longer without getting stuck.

Related Work

Johnson[7] showed simple necessary and sufficient conditions in the case of second degree Bézier triangles in the case where two of the edges were linear. Vavasis [6] showed sufficient conditions for the Bézier elements defined on a more general set of domains. That invertible meshes with Bézier elements could be generated was demonstrated by Boivin et al [1]. A framework for performing finite element method on fluids using Bézier elements was laid out by Miller, et al. [2]

Tumble is a software package built for the purposes of simulating blood flow for the Sangria project. In order to reduce the error introduced, it uses purely Lagrangian finite element method on a moving mesh composed of second degree Bézier triangles. [3]

Background

We define a 2D unit triangle in three dimensions as

$$\Delta^2 = \{(\alpha, \beta, \gamma) \mid 0 < \alpha, \beta, \gamma \ \wedge \ \alpha + \beta + \gamma = 1\}$$

Its boundary is

$$\overline{\Delta^2} = \{(\alpha, \beta, \gamma) \mid 0 \leq \alpha, \beta, \gamma \ \wedge \ (0 = \alpha \vee 0 = \beta \vee 0 = \gamma) \ \wedge \ \alpha + \beta + \gamma = 1\}$$

A Bézier triangle polynomial is a polynomial function $P : \Delta^2 \to \mathbb{R}^2$ where

$$P(x) = \sum_{i=0}^{p} \sum_{j=0}^{p-i} \mathbf{p}_{i,j} x_1^i x_2^j x_3^{p-i-j} \frac{p!}{i!j!(p-i-j)!}$$

The second degree Bézier polynomial has the form

$$P(x) = \mathbf{a}x_1^2 + \mathbf{b}x_2^2 + \mathbf{c}x_3^2 + 2\mathbf{bc}x_2x_3 + 2\mathbf{ac}x_1x_3 + 2\mathbf{ab}x_1x_2$$

Where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{ab}, \mathbf{ac}, \mathbf{ab}$are 2D vectors corresponding to the corners and control points of the triangle respectively.

Global-Invertibility there exists a function $G : R^2 \to \Delta^2$ such that for all $x \in \Delta^2$, $G(P(x)) = x$.

Local-Invertibility the Jacobian of P is non-singular everywhere.

Non-Inverted the determinant of the Jacobian of P is positive everywhere.

The non-inverted condition is useful for finite element analysis as this condition implies that the mesh is globally invertible - for every x in the domain there exists a unique element P in the mesh and a unique $t \in \Delta^2$ s.t. $P(t) = x$. If the mesh does not overlap and the mesh is globally invertible, then it is also the case that each element is non-inverted. If a second degree Bézier triangle is non-inverted, then it is globally invertible.

We can write the Jacobian of a second degree Bézier triangle as

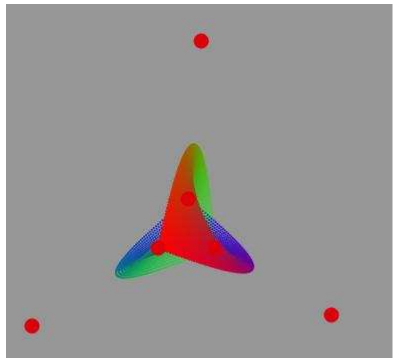
$$\frac{1}{2} J_P(x) = \begin{bmatrix} ab - a \\ bc - a \end{bmatrix} x_1 + \begin{bmatrix} b - ab \\ bc - ab \end{bmatrix} x_2 + \begin{bmatrix} bc - ac \\ c - ac \end{bmatrix} x_3$$

Because we only care if the determinant of the Jacobian is positive and not zero, we can ignore the leading constant.

Pathological Counterexamples

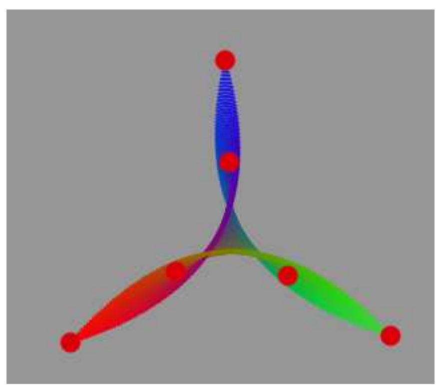
To illustrate the deceiving difficulty in coming up with proper tests for invertibility and loop invariants, we have discovered a number of seemingly obvious conjectures along with their pathological counterexamples.

One misleading conjecture is that the Jacobian being positive along the edges of the domain triangle implies that the Jacobian is positive over the entire triangle. We have a simple counterexample:



In this example, the Jacobian is positive along the edges of the triangle, but negative within the triangle. Note here that the edges are not oriented correctly.

Yet another misleading conjecture is that if the Jacobian is positive along the edges and the boundary is oriented correctly, then the Jacobian is positive over the entire triangle.



In the above example, the Jacobian is positive along the edges, and the boundary is oriented correctly, but the triangle does not have a positive Jacobian everywhere.

The Condition

Let $x \in \Delta^2$. Then let

$$J_P(x) = \begin{bmatrix} -A_1- \\ -B_1- \end{bmatrix} x_1 + \begin{bmatrix} -A_2- \\ -B_2- \end{bmatrix} x_2 + \begin{bmatrix} -A_3- \\ -B_3- \end{bmatrix} x_3$$

where $A_i$and B_i are edge vectors of the Bézier triangle.

$$det(J_P(x)) = F(x) = det(\begin{bmatrix} -A_1- \\ -B_1- \end{bmatrix} x_1 + \begin{bmatrix} -A_2- \\ -B_2- \end{bmatrix} x_2 + \begin{bmatrix} -A_3- \\ -B_3- \end{bmatrix} x_3)$$

$$F(x) = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j (A_{i,1}B_{j,2} - A_{i,2}B_{j,1})$$

Let $Q_{i,j} = A_i \times B_j$

$$F(x) = x^T Q x = x^T \frac{Q+Q^T}{2} x + x^T \frac{Q-Q^T}{2} x$$

The latter term $x^T \frac{Q-Q^T}{2} x$ is identically zero for any matrix Q. So let $Q_0 = \frac{Q+Q^T}{2}$. Q_0 is symmetric. This symmetry is useful for reducing the number of operations applicable.

$$F(x) = x^T Q_0 x$$

For $x \in \Delta^2$ we can write

$$x_3 = 1 - x_1 - x_2$$

Let $y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. We can write

$$\det\left(J\begin{pmatrix} y_1 \\ y_2 \\ 1 - y_1 - y_2 \end{pmatrix}\right) = f(y) = y^T N y + Ly + k$$

Where N is a symmetric 2×2 matrix of rank 2. Thus

$$Df = 2Ny + L$$

Setting this equal to zero we observe

$$Ny = -\frac{L}{2}$$

is satisfied iff y is an extrema. Because $f(y)$ is only a second degree continuous polynomial equation, y satisfying $Ny = \frac{L}{2}$ is the only extrema.

We can solve explicitly for y such that $Df(y) = 0$. If $y \in \Delta^2$ such that $Df(y) = 0$ then if $f(y) < 0$ the triangle is inverted. If $f(y) > 0$, we need to find the minimum of $f(y)$ over the boundary of the triangle.

If $f(y) \geq 0$ ($\forall y \in \overline{\Delta^2}$) then $det(J) \geq 0$ ($\forall y \in \Delta^2$).

Suppose the extrema y of f is not in Δ^2. Then f has no extrema in Δ^2. Again, we only need to find the minimum of f on $\overline{\Delta^2}$. This is equivalent to finding the minimum of three univariate functions.

The routine is simply to calculate the matrix N and L, ensure that the minimum on the edges of reference triangle are positive, and ensure that if the global minimum is within the bounds of the triangle, that it is also positive.

A demonstration of the test can be found at: http://www.andrew.cmu.edu/user/mmirman/bezier/index.html

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