Orthogonal Segment Stabbing*

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Abstract

We study a class of geometric stabbing/covering problems for sets of line segments, rays, and lines in the plane. While we demonstrate that the problems on sets of horizontal/vertical line segments are NP-complete, we show that versions involving (parallel) rays or lines are polynomially solvable.

1 Introduction

We consider some geometric optimal *stabbing* problems for orthogonal (axis-parallel) line segments in the plane. Throughout this paper, we let H denote a given set of (disjoint) horizontal segments and let V denote a given set of (disjoint) vertical segments. We discuss the following problems:

1. In the *Orthogonal Segment Dominating Set* problem (OSDS), we want to find a minimum cardinality subset $S^* \subseteq H \cup V$ such that for each segment $s \in H \cup V$, either $s \in S^*$ or s is stabbed (intersected) by some segment $s' \in S^*$.

2. In the *Orthogonal Segment Covering* problem (OSC), we want to find a subset $S \subseteq V$ of *vertical* segments of minimal size that stab all horizontal segments (assuming such a set exists).

   We study variants of OSC in which segments are extended to rays:

   (a) If V is a set of downward-directed rays, then we refer to the OSC as the *Stabbing Segments with Rays* problem (SSR): Find a minimum-cardinality subset of the vertical rays that stab all horizontal segments H.

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(b) If H is a set of rightward-directed rays, then we refer to the OSC as the *Stabbing Rays with Segments* problem (SRS): Find a minimum-cardinality subset of the vertical segments that stab all horizontal rays H.

(c) If H is a set of rightward-directed rays and V is a set of downward-directed rays, then we refer to the OSC as the *Stabbing Rays with Rays* problem (SRR): Find a minimum-cardinality subset of the vertical rays that stab all horizontal rays H.

We show that, in general, both of the problems OSDS and OSC are NP-complete. For the special cases, SSR, SRS, and SRR, of OSC we obtain polynomial-time solutions, based on dynamic programming.

Our results for OSC are summarized in Fig. 1. The main trend to observe is that the more "endpoints" that specify the elements (V or H), the more difficult the OSC problem is: segments are more difficult than rays, which are more difficult than lines.

Since both OSDS and OSC are instances of set cover problems, they have $O(\log n)$-approximation algorithms based on the greedy set cover heuristic; see, e.g., Hochbaum [9]. We give an $O(n \log n)$ 2-approximation algorithm for the SRS problem.

Our algorithms for the SSR and SRS problems solve, in fact, a more general problem in which the segments are not necessarily horizontal or vertical – we only need that the segments are *non-crossing*.

	$\vert$	$\downarrow$	$\updownarrow$
—	NP-complete	$O(n^3)$	$O(n \log n)$
$\rightarrow$	$O(n^6)$ $O(n \log n)$ 2-approx	$O(n \log n)$	$O(n)$
$\leftrightarrow$	$O(n \log n)$	$O(n)$	$O(1)$

Figure 1: Summary of our results for the OSC problem. Complexities are stated in terms of $n = |V| + |H|$. The columns are labeled according to the type of vertical elements V: segments, downwards rays, or infinite lines. The rows are labeled according to the type of horizontal elements H: segments, rightwards rays, or infinite lines.

Motivation and related work.

The problems we study are fundamental versions of geometric set cover and hitting set problems. Given a set $\mathcal{O}$ of objects and a set P of points such that each object contains at least one point, the objective in the minimum piercing (hitting set) problem is to find a minimum cardinality subset $P^* \subseteq P$ such that every object is pierced by at least one point of P^*. The set cover problem is to find a minimum cardinality subset $\mathcal{O}^* \subseteq \mathcal{O}$ such that each point P is covered by at least one object of $\mathcal{O}^*$. These problems are known to be NP-complete, even if the objects are axis-aligned rectangles in the plane [5, 9]. The SSR (resp., SRS) problem is the special case of the piercing (resp., set cover) problem in which the objects are axis-aligned rectangles whose top edges all lie on a common horizontal line.

In addition to theoretical interest in fundamental geometric covering problems, we are motivated by the following related problems:

(a) Visibility-preserving terrain approximation: Given a polyhedral terrain T and a set P of sample points above T, the goal is to replace T with a minimum-complexity approximation surface, T^*, for which the visibility graph of P relative to T^* is the same as (or very close to the same as) the visibility graph of P relative to T (i.e., two points of P are visible with respect to T if and only if they are visible with respect to T^*). This problem has been introduced recently in the work of Ben-Moshe et al. [3]. If T is a terrain in the plane (i.e., a piecewise-linear function of one variable), and we are required to use a subset of the vertices of T in the approximating terrain T^*, then in order to preserve visibility it is necessary to keep a subset of vertices of T whose downwards rays stab all of the line segments joining pairs of points in P that do *not* see each other (i.e., all *invisible edges* must be stabbed). Thus, we are motivated to study the covering problem in which we must select a minimum-cardinality subset of downwards rays that intersect all of the segments that represent invisibility edges with respect to a terrain. Note that this set of segments may have many crossings; in this paper, we solve the case in which the segments are non-crossing and pose the general case as an open problem.

(b) Minimum-link watchman route: Given a connected network of horizontal and vertical streets, the goal is to compute a watchman route (i.e., a route that allows one to see along every street segment) that has the fewest links (fewest turns); see [1]. For each street segment, a watchman route must include at least one segment that intersects it; thus, this problem is closely related to the OSDS problem.

Easy cases.

The results involving lines (last column and last row of Table 1) are quite straightforward; we include them in the table in order to show the comparison with the segment and ray cases. Consider, for example, the stabbing segments with lines problem (last column). It is easy to see that the greedy algorithm that repeatedly selects the leftmost line that stabs the segment with the rightmost left endpoint, among the remaining segments, finds an optimal subset of lines in time $O(n \log n)$.

The other easy case in Table 1 is that of the SRR problem, which we describe briefly. Let $H = \{r_1, \ldots, r_m\}$ be a set of m horizontal rightwards rays, indexed according to decreasing y-coordinate, and let $V = \{d_1, \ldots, d_n\}$ be a set of n downwards-directed rays, indexed according to increasing x-coordinate. Our goal in the Stabbing Rays with Rays problem (SRR) is to find a minimum cardinality subset $V^* \subseteq V$, if one exists, such that, for each rightwards ray $r_i \in H$, there exists a downwards ray $d_j \in V^*$ that stabs it. The algorithm is very simple: For ray r_1 (the highest rightwards ray), we must select a downwards ray to stab it; we may as well pick the ray, $d^{(1)} \in V$, that is furthest to the right, as it will stab as many rightwards rays of H as possible. Then, we remove from H the set of rightwards rays stabbed by $d^{(1)}$, and repeat, at each step selecting the downwards ray that is furthest to the right (if one exists) that stabs the highest remaining rightwards ray. This algorithm is easily seen to yield an optimal solution (or report that the problem is infeasible). The running time is easily seen to be $O(n \log n)$, based on sorting the rays (sorting H and V by y-coordinate) and using an augmented one-dimensional range tree over the endpoints of the rays in V to identify for a given rightwards ray r_i the endpoint of the downwards ray that lies above it and is furthest to the right.

2 Stabbing Segments with Segments

In this section we prove that OSC and OSDS are NP-complete. Our proofs are based on the following representation theorem due to Ben-Aroyo Hartman et al. [2] and to de Fraysseix et al. [6], and on the corresponding algorithmic result of de Fraysseix et al. [7].

Theorem 1 *[2, 6] Any planar bipartite graph $G = (A, B; E)$ can be represented by a set $I(G)$ of $|A|$ (disjoint) horizontal segments and $|B|$ (disjoint) vertical segments, such that two segments, one horizontal and one vertical, corresponding to vertices a and b, stab each other if and only if $(a, b) \in E$. The set $I(G)$ is said to be a* grid representation *of G.*

Theorem 2 *[7] Let $G = (A, B; E)$ be a planar bipartite graph. One can compute a grid representation $I(G)$ of G in $O(|A \cup B|)$ time.*

2.1 OSDS is NP-Complete

Recall that the Orthogonal Segment Dominating Set problem (OSDS) asks for a minimum cardinality subset $S^* \subseteq H \cup V$ such that for each segment $s \in H \cup V$, either $s \in S^*$ or s is stabbed by some segment s' in S^*.

Theorem 3 *OSDS is NP-complete.*

Proof: We prove that OSDS is NP-complete by a reduction from MINIMUM DOMINATING SET for planar bipartite graphs, whose NP-completeness we establish below. Let $G = (A, B; E)$ be a planar bipartite graph and let $I(G) = V \cup H$ be a grid representation of G. By Theorem 1 and Theorem 2, we know that $I(G)$ exists and can be computed in linear time. This completes the proof of our theorem, since, by definition, OSDS for the segment sets V and H is precisely the MINIMUM DOMINATING SET problem in G.

We were unable to find a reference proving the NP-completeness of MINIMUM DOMINATING SET for planar bipartite graphs; thus, we include the simple proof here. The proof appears implicitly in a paper by Kariv and Hakimi [10]; it is based on the NP-completeness of MINIMUM VERTEX COVER for planar graphs [8].

For a given planar graph G, we cut each edge (u, v) and insert a 4-cycle in the middle of the edge, as shown in Fig. 2. The resulting new graph G' is clearly planar and bipartite; the vertices u, v, x, y belong to the first vertex set and the vertices x, y belong to the second vertex set. Moreover, there exists a vertex cover for G of size k if and only if there exists a dominating set of size $2k$ in G'. □

2.2 OSC is NP-Complete

Recall that the Orthogonal Segment Cover problem (OSC) asks for a minimal subset of a given set V of vertical segments that stabs all of the segments in H, a given set of horizontal segments.

Theorem 4 *OSC is NP-complete.*

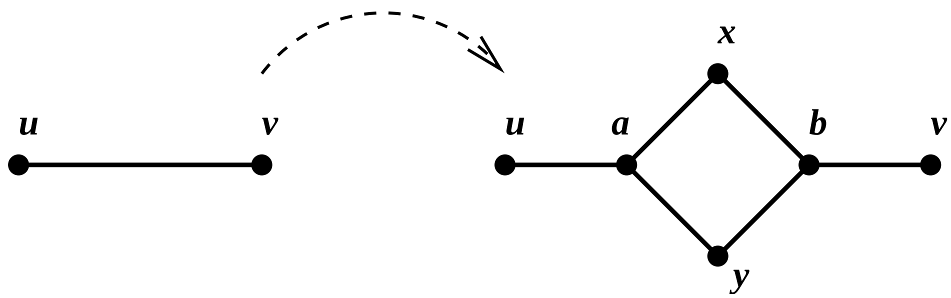

Figure 2: Construction for proof of Theorem 3.

Proof: We prove that OSC is NP-complete by a reduction from MINIMUM VERTEX COVER for planar graphs, which is known to be NP-complete [8]. Let $G = (U, E)$ be a planar graph. By placing a new vertex b_e in the middle of each edge e of E, we obtain a planar bipartite graph $G' = (A, B; F)$, where $A = U$, B corresponds to E, and there is an arc between $a \in A$ and $b \in B$ if and only if a is adjacent to the edge of G corresponding to b. It follows from the construction that G' is planar and bipartite. We compute (in linear time [7]) a grid representation $I(G') = V \cup H$ of G'. This completes the reduction, since a minimum vertex cover for G becomes a minimum subset of A that dominates all vertices in B, which in turn becomes a solution to OSC (i.e., a minimum subset of V that stabs all segments in H). $\square$

3 Stabbing Segments with Rays

Let $H = S = \{s_1, \ldots, s_m\}$ be a set of m horizontal segments and let $V = R = \{r_1, \ldots, r_n\}$ be a set of n downwards-directed rays. We assume that no two rays overlap and that the rays are indexed according to the x-coordinates of the rays, so that r_i is (strictly) left of r_j if $i < j$. We let (x_i, y_i) denote the endpoint of ray r_i. We also define two special downwards rays, r_0 having endpoint $(-\infty, \infty)$, and r_{n+1} having endpoint (∞, ∞).

Our goal in the Stabbing Segments with Rays problem (SSR) is to find a minimum cardinality subset $R^* \subseteq R$, if one exists, such that, for each segment $s \in S$, there exists a ray $r \in R^*$ that stabs it.

We solve the SSR problem using dynamic programming. For $0 \leq i < j \leq n+1$, we define *subproblem* $Q(i, j)$ to be the problem of finding a minimum cardinality subset of R to stab all of the segments, $S(i, j)$, of S that lie strictly within the vertical strip defined by r_i and r_j; i.e., for each $s \in S(i, j)$, the x-coordinate of each endpoint of s lies strictly between x_i and x_j. Refer to Fig. 3. We let $f(i, j)$ denote the optimal objective function value (cardinality of the subset of rays) for subproblem $Q(i, j)$. Our overall objective is to compute $f(0, n+1)$.

We tabulate the values $f(i, j)$ starting with $f(i, i+1)$, for $i = 0, 1, \ldots, n$:

$$f(i, i+1) = \begin{cases} 0 & \text{if } S(i, i+1) = \emptyset \\ +\infty & \text{otherwise} \end{cases}.$$

The justification is that, since there are no rays in the vertical strip between r_i and r_{i+1}, there is no chance to stab the segments $S(i, i+1)$, if this set is nonempty.

We tabulate values of $f(i, i+\Delta)$, for $\Delta = 2, 3, \ldots$, using the following recursion. For any i such that $i+1 < j$ holds:

$$f(i,j) = \min_{k \in K_{ij}} \{1 + f(i,k) + f(k,j)\},$$

where $K_{ij} = \{i < k < j : r_k$ stabs the segment of $S(i,j)$ having maximum y-cordinate$\}$. Since the segment of $S(i,j)$ having maximum y-coordinate is trivially determined in time $O(m)$, and there are $O(n)$ choices for k, the tabulation of all $O(n^2)$ values is readily done in time $O(n^2(m+n))$, using $O(n^2+m)$ space. Correctness of this recursion is established in the following lemma:

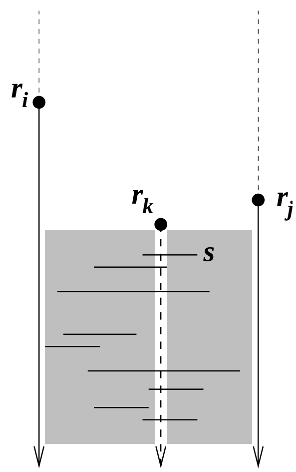

Figure 3: Definition of subproblem for the SSR problem.

Lemma 1 *Any optimal solution for $Q(i,j)$, with $S(i,j) \neq \emptyset$, consists of a ray, r_k ($i < k < j$), stabbing the segment of $S(i,j)$ having maximum y-coordinate, together with the rays in optimal solutions of the two subproblems $Q(i,k)$ and $Q(k,j)$.*

Proof: Let R^* be an optimal solution for subproblem $Q(i,j)$, with $S(i,j) \neq \emptyset$. Let $s \in S(i,j)$ be the segment of $S(i,j)$ having maximum y-coordinate. There must be a ray, $r_k \in R^*$, with $i < k < j$, that stabs s. Any segment in $S(i,j)$ that is *not* stabbed by r_k lies either completely to the left or completely to the right of r_k; thus, $S(i,j)$ is partitioned into $S(i,k)$, $S(k,j)$, and that subset of segments that are stabbed by r_k. Thus, R^* consists of r_k, together with the optimal solutions to the subproblems $Q(i,k)$ and $Q(k,j)$. □

Theorem 5 *The SSR problem for n rays and m segments can be solved exactly in $O(n^2(m+n))$ time using $O(n^2+m)$ space.*

In fact, our algorithm and its analysis apply to a more general problem, in which the rays R are all downward (and thus parallel), but the segments S can be arbitrary non-crossing segments in the plane (not necessarily horizontal). The only change needed in the analysis is the observation that the segments S, as well as the ray endpoints, can be partially ordered according to the "above" relationship. The set K_{ij}, then, is defined to be those rays that stab any one segment of $S(i,j)$ that is maximal according to the partial order.

4 Stabbing Rays with Segments

Let $H = S = \{s_1, \ldots, s_m\}$ be a set of m horizontal segments, indexed according to decreasing y-coordinate, and let $V = R = \{r_1, \ldots, r_n\}$ be a set of n downwards-directed rays, with endpoints (x_i, y_i), indexed according to increasing x_i. We also define a special horizontal segment, $s_0 = ((-\infty, \infty), (\infty, \infty))$, lying above all segments and rays, and two special rays, r_0 having endpoint $(-\infty, \infty)$, and r_{n+1} having endpoint (∞, ∞).

Our goal in the Stabbing Rays with Segments problem (SRS) is to find a minimum cardinality subset $S^* \subseteq S$, if one exists, such that, for each ray $r \in R$, there exists a segment $s \in S^*$ that stabs it.[1]

We solve the SRS problem using dynamic programming. For $0 \leq i < j \leq n + 1$ and $0 \leq k \leq m$, we define $S(i, j, k)$ to be the set of segments that lie below s_k and strictly within the vertical strip defined by r_i and r_j. Similarly, we define $R(i, j, k) \subseteq R$ to be the subset of rays that do not cross s_k and lie strictly within the vertical strip defined by r_i and r_j. Then, we define *subproblem* $Q(i, j, k)$ to be the problem of finding a minimum cardinality subset of the segments $S(i, j, k)$ to stab all of the rays $R(i, j, k)$. We let $f(i, j, k)$ denote the optimal objective function value (minimum cardinality of the subset of segments) for subproblem $Q(i, j, k)$. Our overall objective is to compute $f(0, n + 1, 0)$.

We tabulate the values $f(i, j, k)$. First, observe that $f(i, i + 1, 0) = 0$, for $i = 0, 1, \ldots, n$. (Since there are no rays in the vertical strip between r_i and r_{i+1}, we do not need any segments to stab them.)

Using the following recursion, we tabulate values of $f(i, i + \Delta, k)$, for $\Delta = 2, 3, \ldots$ and for $k = 0, 1, \ldots, m$. For any $i < j$ with $j - i > 1$:

$$f(i, j, k) = \min_{(i', j', l) \in L_{ijk}} \left\{1 + f(i, i', l) + f(i', j', l) + f(j', j, l)\right\},$$

where $L_{ijk} = \{(i', j', l) : i < i' \leq j' < j$, rays $r_{i'}$ and $r_{j'}$ are stabbed by segment s_l, and $s_l \in S(i, j, k)\}$. Since the set L_{ijk} is easily determined within time proportional to its cardinality, $O(mn^2)$, the tabulation of all $O(mn^2)$ values $f(i, j, k)$ is readily done in time $O(m^2 n^4)$, using $O(mn^2)$ space.

For a solution S^* to subproblem $Q(i, j, k)$, we say that a segment $s \in S^*$ *uniquely stabs* a ray $r \in R(i, j, k)$ if no other segment $s' \in S^*$ stabs ray r.

Correctness of this recursion is established in the following lemma:

Lemma 2 *Any optimal solution S^* for $Q(i, j, k)$, with $R(i, j, k) \neq \emptyset$, consists of a highest (maximum y-cordinate) segment, $s_l \in S^*$, which uniquely stabs rays $r_{i'} \in R(i, j, k)$ and $r_{j'} \in R(i, j, k)$ (possibly $r_{i'} = r_{j'}$), the leftmost and rightmost, respectively, rays of $R(i, j, k)$ that are uniquely stabbed by s_l, together with the segments in optimal solutions of the three subproblems $Q(i, i', l)$, $Q(i', j', l)$, and $Q(j', j, l)$.*

Proof: Let S^* be an optimal solution for the subproblem $Q(i, j, k)$, with $S(i, j, k) \neq \emptyset$. Let $s_l \in S(i, j, k)$ be the segment of S^* having maximum y-coordinate. There must be at least one ray of $R(i, j, k)$ that is uniquely stabbed by s_l (among the segments S^*); otherwise, s_l could be deleted from S^* with the remaining segments still stabbing all of $R(i, j, k)$, a

[1] In Section 1, the SRS was introduced in terms of vertical segments stabbing horizontal rays, as a special case of the OSC problem; here, we find it convenient, for consistency with the SSR terminology of the previous section, to define SRS in terms of horizontal segments stabbing vertical rays.

contradiction to the optimality of S^*. Thus, we let $r_{i'}$ (resp., $r_{j'}$) be the leftmost (resp., rightmost) such ray; possibly $i' = j'$ if there is only one ray uniquely stabbed by s_l. The remaining segments of S^* all lie below (in y-coordinate) s_l, and none of them intersects any of the rays r_i, $r_{i'}$, $r_{j'}$, and r_j; thus, the remaining segments of S^* are partitioned among the 3 vertical strips defined by rays r_i, $r_{i'}$, $r_{j'}$, and r_j, and, by optimality, they must constitute solutions to the 3 corresponding subproblems, $Q(i, i', l)$, $Q(i', j', l)$, and $Q(j', j, l)$. □

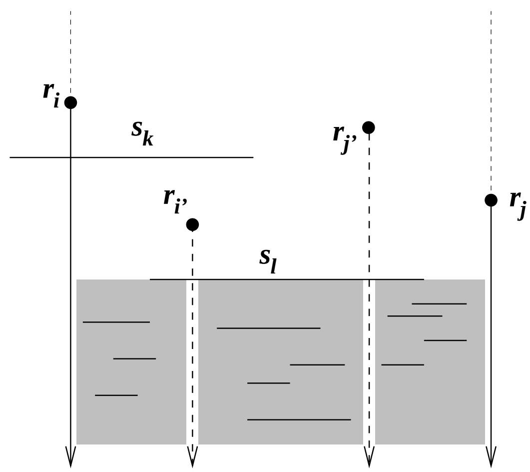

Figure 4: Definition of subproblem for the SRS problem.

Theorem 6 *The SRS problem for n rays and m segments can be solved exactly in $O(m^2n^4)$ time using $O(mn^2)$ space.*

We remark that, as with the SSR problem (Theorem 5), our algorithm and its analysis apply to a more general problem, in which the rays R are all downward (and thus parallel), but the segments S can be arbitrary non-crossing segments in the plane (not necessarily horizontal).

We next show that the SRS problem has a simple 2-approximation algorithm that is significantly more efficient than the dynamic programming solution that solves it exactly:

Theorem 7 *There is a 2-approximation algorithm for the SRS problem for n rays and m segments that runs in time $O((m+n)\log(m+n))$, using $O(n + m\log m)$ space.*

Proof: Let r_i be the ray whose endpoint, (x_i, y_i), has the smallest y-coordinate. (We can assume without loss of generality that no two ray endpoints have the same x- or y-coordinate.) Assuming that there is a feasible solution to the SRS instance, there must exist a segment of S that stabs r_i. We can compute the set $S_i \subseteq S$ of segments that intersect r_i. Among the segments of S_i, let s_{left} (resp., s_{right}) be a segment whose left endpoint is leftmost (resp., right endpoint is rightmost); it may be that $s_{left} = s_{right}$. We add both s_{left} and s_{right} to our heuristic solution set, $\bar{S}$, remove from R all of the rays that are stabbed by s_{left} or s_{right}, and then repeat: find the lowest ray, add to $\bar{S}$ the leftmost/rightmost segments stabbing it, remove the stabbed rays, etc, until $R = \emptyset$. Each iteration requires that (a) we find the lowest ray endpoint from among the set R of remaining unstabbed rays, (b) we identify the set S_i of segments that stab the lowest ray, and (c) we identify and remove from R the rays that are stabbed by s_{left} or by s_{right}. Steps (a) and (c) are readily done within total time $O(n \log n)$, e.g., by maintaining rays sorted by x- and by y-coordinates of endpoints. Step (b) is readily done within time $O(|S_i| + \log m)$, using a segment tree data structure (constructed in time and space $O(m \log m)$) for S, in which each internal node of the tree stores the set of segments that span the associated canonical x-interval, sorted by y-coordinate. (A standard "layering" technique allows the search in y during a query to be done in time $O(1)$ per node of a root-to-leaf path, after an $O(\log m)$ search at the root; see [4].) Thus, the overall running time of the algorithm is $O((m + n) \log(m + n))$.

The number of segments, $|\bar{S}|$, in our heuristic solution is at most 2 times the number of iterations in the algorithm. An optimal solution must have at least as many segments as there are iterations of the algorithm, since no segment of S stabs both of the (lowest) rays, r_i and r_j, identified in distinct iterations of the algorithm. □

5 Conclusion

We conclude with some open questions. First, what is the complexity of the SSR problem if the segments S may arbitrarily intersect? Our algorithm critically uses disjointness of the segments; is the general case NP-complete?

Second, using techniques similar to those presented here for the OSC problem, we expect that a table of results like Table 1 can be obtained for the dominating set problem (OSDS).

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