Approximating Watchman Routes*

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Abstract

Given a connected polygonal domain P, the watchman route problem is to compute a shortest path or tour for a mobile guard (the "watchman") that is required to see every point of P. While the watchman route problem is polynomially solvable in simple polygons, it is known to be NP-hard in polygons with holes.

We present the first polynomial-time approximation algorithm for the watchman route problem in polygonal domains. Our algorithm has an approximation factor $O(\log^2 n)$. Further, we prove that the problem cannot be approximated in polynomial time to within a factor of $c \log n$, for a constant $c > 0$, assuming that P≠NP.

1 Introduction

A classic problem in computational geometry is the *watchman route problem* (WRP): Compute a shortest path/tour within a polygonal domain (polygon with holes) P so that every point of P is seen from some point of the path/tour, i.e., compute a shortest "visibility coverage tour". The WRP models a natural problem in robotics, in which a mobile robot/camera is to do a visual inspection of a domain or a part, whose geometry is given. Here, we focus on the *offline* setting, in which the geometry of the domain P is given; the problem has been studied extensively, both in the offline and online setting (see the surveys [24, 25]).

In this paper we give the first polynomial-time approximation algorithm for the WRP in polygonal domains in the plane. Our approximation factor is $O(\log^2 n)$; we also give the first hardness of approximation result for the WRP, showing that the problem cannot be approximated in polynomial time to within a factor of $c \log n$, for a constant $c > 0$, assuming that P≠NP. Here, n is the number of vertices of P.

1.1 Prior and Related Work. While the problem had arisen earlier in robotics, the WRP was first studied from an algorithmic perspective by Chin and Ntafos in 1986 [6], who proved that the problem is NP-hard in general (see [14] for a new proof that addresses gaps in the original proof [6, 7]). Exact algorithms are known for the WRP in simple polygons (domains with no holes) [4, 8, 11, 20, 31, 32]; the best known time bounds [11] are $O(n^3 \log n)$ for the "anchored" WRP, in which the tour is required to pass through a specified anchor point, and $O(n^4 \log n)$ for the "floating" WRP, in which no anchor point is specified. Linear-time algorithms are known for approximating the WRP in simple polygons (with approximation factor $\sqrt{2}$ for the anchored case and approximation factor 2 for the floating case – see [28, 29, 30]) and for exactly solving the special case in which P is a simple rectilinear polygon [7].

Another special case of the WRP that has an exact polynomial-time algorithm is that in which P is the union of a finite set of infinite lines in the plane (i.e., the WRP on a line arrangement) or a set of chords with respect to a simple closed curve [13]; a polylog approximation is also given for the case of WRP on an arrangement of line segments, obtained by a direct mapping of the problem to the group TSP problem.

For the general case, in which P is multiply connected (has holes), very little has been known about the approximability of the WRP; the question of the existence of a constant-factor approximation algorithm for WRP has been posed (e.g., [29]). We know of only one provable approximation algorithm [22], which gives an $O(\log n)$-approximation if P is rectilinear (with holes) and the visibility model is also special ("rectangle visibility", in which p sees q if and only if the rectangle with corners p and q lies fully within P; the requirement of rectangle visibility was not specified in the original conference paper); it does not apply to the general WRP, for ordinary visibility within a polygon domain, as we address here. One can show that the minimum length of a watchman tour for a polygon P with h holes is $O(per(P) + \sqrt{h} \cdot diam(P))$, and this bound is tight for polygons P with $per(P) > c \cdot diam(P)$, for any fixed $c > 2$; see [10, 14] for two different proofs. Dumitrescu and Tóth [14] show how to find a tour of length $O(per(P) + \sqrt{h} \cdot diam(P))$ in time $O(n \log n)$ and also provide related results for polyhedra with holes in 3-space. However, these results do not imply an approximation algorithm, since neither $per(P)$ nor $diam(P)$ are lower bounds for the WRP in a particular P.

The WRP is related to a geometric traveling salesperson problem (TSP), namely the "TSP with neigh-

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borhoods" (TSPN) [2], in which one seeks a shortest tour to visit a set of geometric regions (the neighborhoods). The geometric TSP has a PTAS ([3, 23]), as does the TSPN in certain special cases (e.g., if the regions are "fat" and (weakly) disjoint [12, 16, 26]). Further, there is a quasipolynomial-time approximation scheme (QPTAS) for geometric instances of TSPN in any fixed dimension, and metric spaces of bounded doubling dimension, for the case of fat, (weakly) disjoint regions [5]. The TSPN for arbitrary connected regions in the plane has an $O(\log n)$-approximation [18, 22]; if the regions are (weakly) disjoint, there is a recent $O(1)$-approximation [27]. The WRP can be viewed as the TSPN in the case that the set of regions are the visibility polygons, $VP(p)$, associated with *every* point p of the domain P; thus, the regions are not necessarily fat, they are not disjoint, and the number of regions is uncountably infinite. Further, the WRP is a TSPN instance *with obstacles*, since the tour is required to stay within the region P.

The WRP and TSPN are related also to the *group (or "one-of-a-set") Steiner tree problem* (and the group TSP), in which one is given an undirected graph $G = (V, E)$ with weighted edges and n vertices, and k subsets of V (called *groups* of vertices), and one must find a minimum-weight tree that has at least one vertex from each of the groups. The best approximation bound is obtained by Fakcharoenphol et al. [15], who apply their method of approximating arbitrary metrics by tree metrics to the $O(\log k \log n)$-approximation algorithm for trees given by Garg, Konjevod, and Ravi [17]; the result is an $O(\log^2 n \log k)$-approximation algorithm for the group Steiner tree problem in metric spaces. Halperin and Krauthgamer [19] show that the problem cannot be approximated within ratio $O(\log^{2-\epsilon} k)$ for any $\epsilon > 0$.

1.2 Our Contribution, Outline of Our Approach. Our main result is the first approximation algorithm for the general watchman route problem in the plane. To achieve this, we have to overcome at least two main difficulties:

(1) The continuum – the watchman is required to see *every* point of a continuous two-dimensional domain, making the TSPN methods not directly applicable, since the number of regions that must be visited is uncountable. This difficulty arises also in attempts to approximate the minimum guard problem in polygons: We know of no polynomial-size "sufficiency set" for searching for guards, and, thus, there are no nontrivial approximations known for guarding using any point in the continuum of a polygon or polygonal domain. A similar situation arises for the WRP, since vertices of an optimal tour may be "reflection" points along construction lines that are not passing through a pair of vertices of P; indeed, we know of no polynomial-size set of points within P that suffice for vertices of an optimal tour. See Figure 1.

(2) The obstacles – the corresponding TSPN problem is not in the Euclidean metric, but rather in the geodesic metric induced by the domain P. While many instances of TSPN have constant-factor approximations, or even a PTAS, the presence of obstacles makes the approximation significantly more difficult. In fact, we prove that the TSPN with obstacles as well as the WRP has no polynomial-time approximation better than logarithmic, assuming P≠NP.

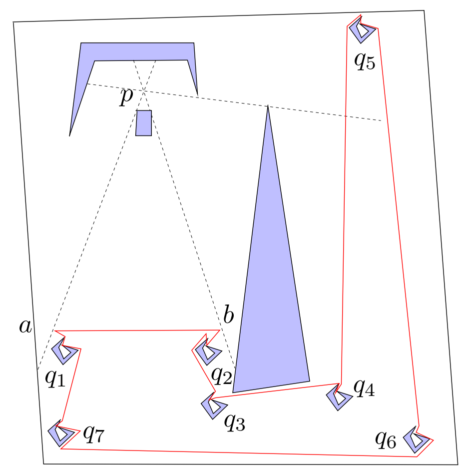

Figure 1: An example WRP tour (red) with reflection points a, b that allow the tour to "just" see point p. The "tiny caves" $q_1, \ldots, q_7$ are holes of P; the tour must visit each cave, but must also make a slight detour to points a and b to be able to see all of P.

We overcome challenge (1) by showing how to localize (approximately) and discretize the domain. In particular, we first enumerate the minimal squares, B, each of which is minimal in size, while seeing all of P that lies outside of it. We show that any optimal solution whose bounding box contains a particular minimal square B must lie within a larger square centered on B, larger by a factor $O(n)$. This allows us to discretize the portion of P that lies inside the larger square, and focus on solving discrete optimization problems. (At this point, we could apply the known group TSP results to obtain a polylogarithmic approximation bound; however, our methods exploit geometry to obtain an even better (by a logarithmic factor) approximation bound.)

We then address challenge (2) by partitioning the problem into two new problems, each of which may be of interest in their own right: the "outer watchman route problem" (OWRP) and the "inner watchman route problem" (IWRP). The OWRP seeks a tour that sees all of P that is outside the tour, while the IWRP takes as input a given simple cycle in P and asks that we augment it with a minimum-length network in order that the augmented network sees all of P that lies within the given cycle.

We show that the OWRP has an efficient approximation algorithm (in fact, a PTAS). To do so, we show that it can be solved exactly in polynomial time, using dynamic programming, if there is a given polynomial-size set of points and we require that the tour be polygonal on this set of points.

We give an approximation to the IWRP, with input cycle given by our solutions to the OWRP for each choice of B, by formulating it as a weighted set cover problem on a discrete set of "geodesic triangles"; we prove that this results in an $O(\log^2 n)$-approximation for the IWRP.

We then prove that we can produce a single tour with the claimed approximation bound, $O(\log^2 n)$, for the original WRP instance. We note that this approximation factor is somewhat better than that known for the group Steiner tree problem in graphs; we attribute this to the fact that we are exploiting geometry.

We also provide hardness of approximation, from Set Cover, showing that the WRP cannot be approximated in polynomial time to within a factor of $c \log n$, for a constant $c > 0$, assuming that P≠NP.

2 Preliminaries

We let ∂X denote the boundary and let $diam(X)$ denote the diameter of a set $X \subset \Re^2$. We abuse notation slightly by using "$|\cdot|$" in two different ways: for a curve (or tree, or one-dimensional manifold) γ, $|\gamma|$ denotes the (Euclidean) length of γ, and for a finite set, S, $|S|$ denotes the cardinality of S. The meaning should be clear from the context.

A *polygonal domain*, P, in the plane ($\Re^2$) is a connected subset of the plane whose boundary is the union of a finite number, n, of (noncrossing) line segments; i.e., P is a *polygon with holes*. The boundary of P consists of an *outer boundary* polygonal cycle, separating P from the unbounded face at infinity, and a set of h *inner boundary* polygonal cycles. Let V_P denote the set of vertices of P. Here, h denotes the number of holes (the genus of P), and $n = |V_P|$ is the total number of vertices of P. (Equivalently, n is the total number of edges of P.)

We let $\pi_P(s,t)$ denote a *(geodesic) shortest path* from $s \in P$ to $t \in P$; i.e., $\pi_P(s,t)$ is a shortest path from s to t that is constrained to stay within the domain P . (In general, there is not a unique shortest path from s to t.) Given a finite point set $S \subset P$, the *geodesic convex hull of* S, denoted $GCH(S)$, is the minimal subset of P that contains S and is closed under taking shortest paths (i.e., for every $s,t \in GCH(S)$, every shortest path between s and t is contained in $GCH(S)$). The boundary of $GCH(S)$ is the boundary of the face at infinity in the arrangement of all shortest paths between pairs of points of S. Shortest paths, as well as geodesic convex hulls, can be computed efficiently (in polynomial time); see [24, 25].

We say that point $p \in P$ *sees* (or is *visible from*) point $q \in P$ if the line segment pq is a subset of the domain P. The *visibility region*, $VP(p)$, of a point $p \in P$ is the (polygonal) set of all points of P that p sees; i.e., $VP(p) = \{q \in P : pq \subseteq P\}$. The visibility region, $VP(X)$, of a subset, $X \subseteq P$, is the set of all points of P that are seen by some point of X; equivalently, $VP(X) = \bigcup_{p\in X} VP(p)$. A tour/tree $T \subseteq P$ is a *visibility covering tour/tree* for P if $VP(T) = P$, i.e., if every point of P is seen by some point of the tour/tree T.

The *watchman route problem* (WRP) is to compute a minimum-length visibility covering tour for P. The *watchman tree problem* (WTP) is to compute a minimum-length visibility covering tree for P. Note that any visibility covering tree can be converted into a visibility covering tour, of length twice that of the tree, by walking around the tree, traversing each edge exactly twice. Thus, for purposes of approximation, the WRP and WTP are equivalent up to a constant factor.

For a simple closed (Jordan) curve $\gamma \subset P$, we let P_γ denote the simply connected subset of $\Re^2$ bounded by γ; if γ is polygonal (piecewise-linear with a finite number of vertices), then P_γ is a simple polygon. We say that a simple closed curve $\gamma \subset P$ is *outer-illuminating* for P if all points of P outside of the region P_γ are seen by γ. More generally, we say that a region $X \subset \Re^2$ is *outer-illuminating with respect to* P if all points of P outside of X are seen by $X \cap P$. (Note that X need not be a subset of P.) Most often, we will be dealing with the case of outer-illuminating squares, B, when $X = B$ is an axis-parallel square in the plane. Similarly, we say that X is *inner-illuminating with respect to* P if all points of $P \cap X$ are seen by the boundary, ∂X, of X.

For a given polygonal domain P, the *outer watchman route problem* (OWRP) is that of computing a shortest outer-illuminating simple closed curve within P. In the *discrete outer watchman route problem* (Discrete-OWRP), we are given also a finite point set

$S \subset P$, with $S \supseteq V_P$, and we must find a shortest simple closed curve, γ, within P so that all of P outside of γ is seen by γ *and* such that γ is a polygonal curve with vertices among the set S.

For a given polygonal domain P, and a given simple closed polygonal curve $\gamma \subset P$, the *inner watchman route problem* (IWRP) is that of computing a minimum-length connected network, $\mathcal{N} \supset \gamma$, such that all of P within γ (i.e., all of $P_\gamma \cap P$) is seen by $\mathcal{N}$. Observe that an optimal solution for IWRP is a network consisting of a union of γ and a finite set of pairwise-disjoint trees (i.e., a forest), each within the region bounded by γ and each connected to γ at a single point. (If a cycle were formed, then an edge could be deleted, shortening the network while still maintaining visibility coverage.)

For the given domain P, an outer-illuminating square, B, is *minimal* if there is no outer-illuminating square, B', with B' a strict subset of B (i.e., with $B' \subset B$). We then refer to B as a *minimal outer-illuminating square* (MOIS) for P.

The following lemma follows from standard critical placement arguments.

LEMMA 2.1. *There are a polynomial (in n) number of minimal outer-illuminating squares associated with an n-vertex polygonal domain P.*

Proof. This follows from standard critical placement arguments, since a square has three degrees of freedom. Briefly, an outer-illuminating square, Σ, can be shrunk, and then translated, until three critical contacts occur, at which Σ is just in contact with $VP(p_1)$, $VP(p_2)$, and $VP(p_3)$, for three (not necessarily distinct) points $p_1, p_2, p_3 \in P$.

If $p_1 \neq p_2, p_3$, we claim that $p_1 \in \partial P$, so that the critical contact is defined by Σ being in contact with an edge extension of P (an edge of P extended through the interior of P to form a chord). Assume, to the contrary, that p_1 is interior to P. Now, Σ is just in contact with $VP(p_1)$ either because (1) a corner of Σ is touching a window edge of $VP(p_1)$ or because (2) an edge of Σ is in contact with the endpoint of a window edge of $VP(p_1)$. In either case, let v_1 be the vertex of P determining the window edge from p_1, and let ℓ_1 be the line through p_1 and v_1. Consider an arbitrarily small neighborhood, $B_\epsilon(p_1) \subset P$, of p_1. Consider a point $q_1 \in B_\epsilon(p_1)$ that lies just off the line ℓ_1, on the same side as the obstacle supporting at v_1. Then, the window defined by q_1 and v_1 fails to intersect Σ; further, $\epsilon > 0$ can be chosen small enough so that q_1 can be chosen in $B_\epsilon(p_1)$ so that q_1 does not see Σ at all, contradicting the fact that Σ is outer-illuminating.

If $p_1 = p_2 \neq p_3$, then $VP(p_1)$ contacts Σ at two distinct places. Let v_1 and v_2 be the (distinct) vertices of P corresponding to the windows of $VP(p_1)$ in contact with Σ. We claim that $p_1 \in \partial P$. Assume to the contrary that p_1 is interior to P. Again we consider a small enough neighborhood, $B_\epsilon(p_1) \subset P$ and observe that there must be a point $q_1 \in B_\epsilon(p_1)$ such that q_1 is not seen by Σ (e.g., within the convex cone defined by the rays $\rho_1 = p_1v_1$ and $\rho_2 = p_1v_2$), contradicting the outer-illumination property of Σ.

If $p_1 = p_2 = p_3$, so that p_1 is a triple-pinned point, with three contacts between Σ and $VP(p_1)$, then we argue similarly to above, exploiting the fact that the three rays, $\rho_1 = p_1v_1$, $\rho_2 = p_1v_2$, and $\rho_3 = p_1v_3$, lie within a convex cone (since they all meet the (convex) square Σ), implying the existence of a pair of rays, one supported by an obstacle on the left, one supported by an obstacle on the right, that bound a convex cone intersecting Σ, within which a point $q_1 \in B_\epsilon(p_1)$ can be chosen so that q_1 does not see Σ. Thus, we again get a contradiction to the outer-illumination property of Σ.

In summary, a minimal outer-illuminating square, Σ, must be in contact with three extension edges of P, and these three edges are enough to determine the square. □

3 Localization and Discretization

In this section, we describe how we utilize the minimal outer-illuminating squares to localize and discretize our watchman route problems.

LEMMA 3.1. *Let γ^* be an optimal solution to the WRP in domain P. Let B be a minimal outer-illuminating square (MOIS) within the axis-aligned bounding box of γ^*. (Such a MOIS must exist, since γ^*, and therefore its bounding box, is outer-illuminating.) Then, $|\gamma^*| = O(n \cdot diam(B))$, implying that γ^* lies within a square, $\bar{B}$, of side length $O(n \cdot diam(B))$, concentric with B.*

Proof. First, by definition of a MOIS, B illuminates all of P outside of B; in particular, the portion, $\partial B \cap P$, of B's boundary within P illuminates all of P outside of B. Now consider illuminating that part of P within B. Consider a vertical decomposition of $B \cap P$ (obtained by extending maximal vertical chords within $B \cap P$, through each vertex of P within B); the decomposition consists of $O(n)$ (convex) trapezoid faces, each of diameter $O(diam(B))$. The edge set, E, of all trapezoids of the decomposition is a connected planar straight-line graph and includes $\partial B \cap P$. (Connectivity follows from the connectivity of the planar dual, which in turn follows from the fact that P is (multiply) connected.) Thus, E illuminates all of $B \cap P$ (since any point $p \in B \cap P$ lies within a trapezoid, whose boundary is illuminated), and also illuminates all of P that is outside B. Further, since there are $O(n)$ faces, each of diameter $O(diam(B))$, we know that the total length, $|E|$, of edges E is $O(n \cdot diam(B))$. Thus, an optimal solution has length $|\gamma^*| \le |E| = O(n \cdot diam(B))$ and must, therefore, stay within the enlarged square, $\bar{B}$, of size $O(n \cdot diam(B))$, concentric with B. □

For a given MOIS, B, we refer to the square $\bar{B}$ from the proof of Lemma 3.1, concentric with B and of size $O(n \cdot diam(B))$, as the *enlargement* of B. Lemma 3.1 tells us that if B is any MOIS within the bounding box of an optimal tour γ^*, then we have approximately localized γ^* (it passes near B) and know its length approximately: γ^* lies within the enlargement, $\bar{B}$, and has length between $\Omega(diam(B))$ and $O(n \cdot diam(B))$.

Using this approximate localization in terms of a MOIS, B, we define a partitioning, $\mathcal{G}_B^\epsilon$, of P into convex cells, such that, for approximation purposes (within factor $(1+\epsilon)$), it will suffice to search for tours (for WRP, OWRP, and IWRP) whose vertices come from the polynomial-size set of vertices of $\mathcal{G}_B^\epsilon$. Specifically, $\mathcal{G}_B^\epsilon$ is defined to be the vertical decomposition of P (into trapezoids, using vertical maximal chords through each vertex of P), refined by an overlay of a regular square grid partitioning $\bar{B}$ (of size $O(n \cdot diam(B))$) into pixels of size $\epsilon \cdot diam(B)/n^2$. Thus, the grid $\mathcal{G}_B^\epsilon$ has a polynomial complexity, $O((n^3/\epsilon)^2)$.

Our discretization lemma below (Lemma 3.3) utilizes the following fact about the structure of optimal solutions.

Lemma 3.2. *An optimal solution, γ^*, for the WRP (or the OWRP) is a polygonal tour, with $O(n^2)$ vertices. Also, an optimal solution for the IWRP is a polygonal network (forest of trees each attached to the input cycle γ at a single point), having $O(n^2)$ vertices.*

Proof. We give the proof of the claim for the WRP; the proofs for the OWRP and IWRP are essentially the same.

Consider the visibility arrangement obtained by overlaying the visibility polygons, $VP(v_i)$, of all vertices of P. (Alternatively, the arrangement is defined by the visibility graph, with maximally extended edges.) Consider any single (convex) cell, $\sigma \subset P$, in the arrangement. (There are $O(n^4)$ cells in the arrangement of $O(n^2)$ segments that make up the windows of all polygons $VP(v_i)$.) For all points $p \in \sigma$, the visibility polygon $VP(p)$ has constant combinatorial type; its boundary consists of portions of ∂P, together with window edges defined by vertices v that are visible from p, and the extension of the segment pv to yield a maximal chord. The intersection of γ^* with σ is a discrete (possibly countably infinite) set of curves, $\gamma_1, \gamma_2, \ldots$. As $p \in \gamma^* \cap \sigma$ varies along a curve γ_i, its visibility polygon $VP(p)$ varies continuously in a very controlled way – the windows pivot about their defining vertices v. Thus, associated with each vertex v visible from σ, there is an *extreme point*, $p_v \in \gamma^* \cap \sigma$, which maximizes the visibility with respect to v, pivoting the window as far as possible in the clockwise (resp., counterclockwise) direction if v supports the window on the right (resp., left). See Figure 2(top). The union of the visibility polygons $VP(p_v)$ over vertices v visible from σ is therefore equal to the union of visibility polygons $VP(p)$ over all points $p \in \gamma^* \cap \sigma$. Thus, $\gamma^* \cap \sigma$ can be replaced by a single polygonal curve within σ that visits all of the extreme points p_v. Specifically,, γ_1 can be replaced by a (polygonal) TSP path, having the same endpoints (along $\partial\sigma$) as γ_1, through the extreme points within σ, while all other curves γ_i, $i \ge 2$, are replaced by single segments joining their entry points to exit points. This replacement preserves the visibility coverage of $\gamma^* \cap \sigma$, while making $\gamma^* \cap \sigma$ polygonal. In fact, if we consider the set of all extreme points of γ^*, over all cells σ, we obtain a set of $O(n^5)$ extreme points in total ($O(n)$ per cell of the arrangement), and any polygonal tour visiting all of these extreme points suffices to see all of P. Thus, since γ^* is the shortest possible tour seeing all of P, γ^* must be a TSP tour of the extreme points, and thus γ^* must be polygonal.

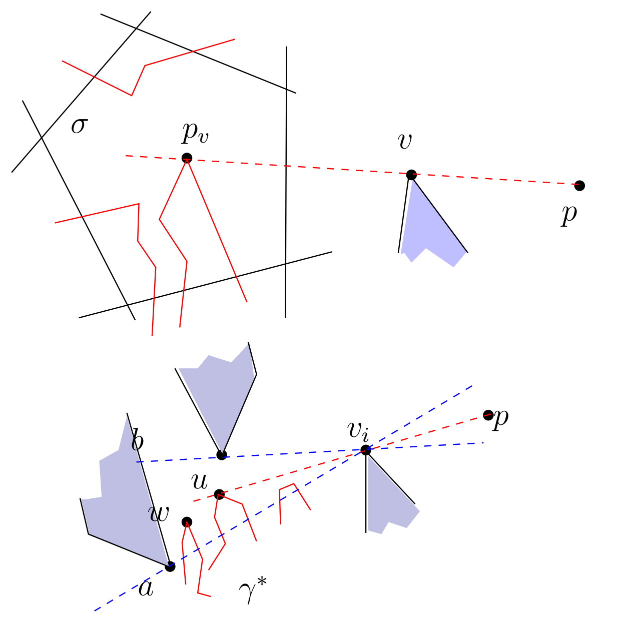

Figure 2: Proof of Lemma 3.2

The above argument gives a rather weak bound on the number of vertices of γ^*. To obtain a better bound of $O(n^2)$, we argue as follows. We know that γ^* is polygonal; it has two types of vertices – those that are vertices of P and those that are not. The vertices of γ^* that are not vertices of P must be *reflection points* that lie interior to window edges; these are the extreme points in the argument above. Note that local optimality implies that the angle of incidence equals the angle of reflection when γ^* enters and leaves a reflection point. For each visibility polygon $VP(v_i)$ corresponding to a vertex $v_i \in V_P$, we can partition $VP(v_i)$ into *visibility triangles*, $\Delta v_i ab$, where $ab \subset \partial VP(v_i)$ and a and b are vertices of the star-shaped polygon $VP(v_i)$. (Assuming no three vertices of P are collinear, if v_i sees k_i vertices of P, then there are exactly k_i visibility triangles $\Delta v_i ab$ incident to v_i.) We claim that within each visibility triangle $\Delta v_i ab$ there can be at most one reflection point of γ^* that reflects off of a window incident to v_i (i.e., off of a window edge bounding $VP(p)$, for some p that sees v_i, having corresponding window edge within the triangle $\Delta v_i ab$). If, instead, γ^* had two or more reflection points within $\Delta v_i ab$, with respect to windows incident on v_i, then one of them (say, u) dominates the other one (say, w), in that the connected component of $\gamma^* \cap \Delta v_i ab$ that visits u must cross the window incident to v_i that passes through w. See Figure 2(bottom). Thus, γ^* can be locally shortened by clipping off w, a contradiction to its optimality. Thus, γ^* has at most one reflection point with respect to v_i (i.e., on a window incident to v_i) per visibility triangle $\Delta v_i ab$. Overall, there are $O(n^2)$ visibility triangles, so γ^* has $O(n^2)$ vertices. □

Remark: While it suffices for our purposes to show a polynomial bound on the number of vertices of an optimal solution γ^*, we believe that the quadratic bound of Lemma 3.2 is weak; we conjecture that γ^* has complexity $O(n)$. (There are simple examples in which γ^* has complexity $\Omega(n)$ within an n-vertex domain P.)

LEMMA 3.3. *Let γ^* be an optimal solution to the WRP and let $\epsilon > 0$ be fixed. We can compute, in polynomial time, a polynomial-size set, $\mathcal{G}$, of points within P such that there exists a polygonal tour, $\gamma_{\mathcal{G}}$, with vertices among $\mathcal{G}$, whose length is at most $(1+\epsilon)|\gamma^*|$. The same statement holds for the OWRP/IWRP.*

Proof. We begin with the WRP. Let B be a MOIS within the bounding box of γ^*, and let $\bar{B}$ be the associated enlarged square of size $O(n \cdot diam(B))$, centered on B. Let $\mathcal{G}$ be the vertices of $\mathcal{G}_B^\epsilon$, the partitioning of P defined above. We do not know B in advance, so our algorithm will iteratively try each choice of MOIS, B. (Alternatively, we can compute the partitionings $\mathcal{G}_B^\epsilon$ for each choice of B, and then define $\mathcal{G}$ to be one large set of all vertices in all of these partitionings.)

We now show how to replace γ^* with $\gamma_{\mathcal{G}}$, having vertices among $\mathcal{G}$. For each edge $u_i u_{i+1}$ of γ^*, we "encase" the edge in a simple polygon, Q_i, whose vertices are in $\mathcal{G}$. Specifically, let σ_i and σ_{i+1} denote the (convex) cells of the partition $\mathcal{G}_B^\epsilon$ that contain u_i and u_{i+1}, respectively. If $\sigma_i = \sigma_{i+1}$, then we replace $u_i u_{i+1}$ with the cycle $\partial\sigma_i$ (whose vertices are from $\mathcal{G}$, by definition). If $\sigma_i \neq \sigma_{i+1}$, then we let $u_i' \in \partial\sigma_i$ denote the point where $u_i u_{i+1}$ exits σ_i (through its bounding edge $a_i b_i$, oriented clockwise around σ_i), and let $u_{i+1}' \in \partial\sigma_{i+1}$ denote the point where $u_i u_{i+1}$ enters σ_{i+1} (through its bounding edge $a_{i+1} b_{i+1}$, oriented clockwise around σ_{i+1}). Now, consider the cycle that starts at grid vertex $a_i \in \partial\sigma_i$, goes along $a_i b_i$ to u_i', then goes along $u_i' u_{i+1}'$ to u_{i+1}', then goes clockwise around $\partial\sigma_{i+1}$ to b_{i+1}, a_{i+1}, and back to u_{i+1}', then back along $u_{i+1}' u_i'$ to u_i, then back around $\partial\sigma_i$ to the starting point, a_i. We "pull taut" (with respect to P) the two subpaths $(a_i, u_i', u_{i+1}', b_{i+1})$ and $(a_{i+1}, u_{i+1}', u_i', b_i)$, resulting in polygonal paths that bend at vertices of P, resulting in an overall polygonal cycle, C_i, that encloses segment $u_i u_{i+1}$ and has vertices among $\mathcal{G}$. Further, the length of C_i is at most $2|u_i u_{i+1}| + O(\epsilon \cdot diam(B)/n^2)$, since the sizes of cells σ_i and σ_{i+1} are $O(\epsilon \cdot diam(B)/n^2)$. Note too, by construction, that C_i does not surround any holes of P. Refer to Figure 3.

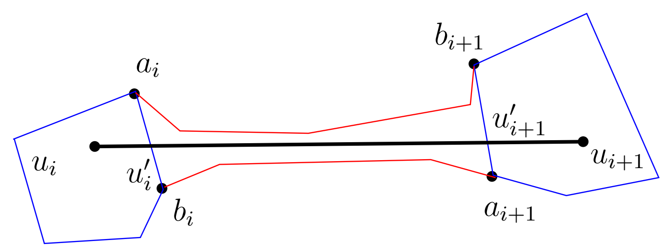

Figure 3: Proof of Lemma 3.3

Now, all points of P that are inside of C_i are seen by C_i, and all points of P outside of C_i that are seen by $u_i u_{i+1}$ are seen by C_i (by the Jordan Curve Theorem, since C_i is a simple closed curve). Thus, the cycle C_i sees every point of P that the edge $u_i u_{i+1}$ does. Thus, taking the union of all such cycles C_i over all edges of γ^* results in a connected polygonal network whose vertices are among the grid points $\mathcal{G}$. By summing the bounds on the lengths of the cycles C_i, over all $O(n^2)$ edges of γ^*, we get that the length of the network is at most $2|\gamma^*| + O(\epsilon \cdot diam(B))$, which is at most $(2+\epsilon')|\gamma^*|$, for an appropriate choice of $\epsilon' > 0$ (since $|\gamma^*| \geq diam(B)$).

In fact, for visibility coverage it suffices to remove from each cycle C_i one of the two taut subpaths, say the one from a_{i+1} to b_i. To see this, note that any line-of-sight segment joining a point $p \in P$ to edge $u_i u_{i+1}$ must either cross the remaining portion of cycle C_i or can be extended to meet with the remaining portion of C_i. Thus, the total length of the resulting connected graph we obtain is at most $(1+\epsilon)|\gamma^*|$.

For the OWRP/IWRP, the same strategy holds for obtaining a grid-rounded solution. □

4 The Outer Watchman Route Problem (OWRP)

We turn now to the outer watchman route problem, for which we will give an exact solution for the discrete version. Combining this with our discretization from Lemma 3.3, we obtain a PTAS for OWRP.

LEMMA 4.1. *An optimal solution to the Discrete-OWRP with respect to P is a polygonal curve, γ^*, bounding a simple polygon P_{γ^*}, with the following properties: (a). the region P_{γ^*} is geodesically convex with respect to P; and (b). γ^* consists of a union of shortest paths within P, $\pi_P(p_1,p_2), \pi_P(p_2,p_3), \ldots, \pi_P(p_{h-1},p_h), \pi_P(p_h,p_1)$.*

Proof. By the definition of the Discrete-OWRP, γ^* has vertices among S.

For claim (a), consider the geodesic convex hull, $GCH(\gamma^*)$, of the vertices (points of S) that appear on γ^*. Since $P_{\gamma^*} \subseteq GCH(\gamma^*)$, by definition of geodesic convex hull, we know that the boundary of $GCH(\gamma^*)$ also sees all of the points of P outside of it. More formally, this follows from the Jordan Curve Theorem, since any point $p \in P$ outside of $GCH(\gamma^*)$ can see γ^*, and $\partial GCH(\gamma^*)$ is a simple closed curve separating p from γ^*.

Claim (b) follows from claim (a) and the definition of geodesic convex hull. □

THEOREM 4.1. *The Discrete-OWRP within a polygonal domain P, and for a given input set $S \supseteq V_P$, can be solved exactly in polynomial (in $|S|$) time.*

Proof. We give a dynamic programming algorithm. We let $n = |S|$ in this proof; $|V_P| \le n$.

For each $p_i \in S$, we compute the tree, $SPT(p_i)$, of shortest paths within P from p_i to all points of S. This can be done in time $O(n \log n)$, where n is the total number of vertices of P and points of S [21]. The overlay of these trees forms an arrangement of line segments of total complexity $O(n^4)$.

We define the *wedge*, W_{ijk}, associated with points $p_i, p_j, p_k \in S$, to be the subset of P consisting of points $p \in P$ for which the shortest path $\pi_P(p_j,p)$ lies strictly to the right (clockwise) of the shortest path $\pi_P(p_j,p_i)$ *and* strictly to the left (counterclockwise) of the shortest path $\pi_P(p_j,p_k)$. (For points $p,q,r \in P$, we say that shortest path $\pi_P(p,q)$ is to the right (resp., left) of shortest path $\pi_P(p,r)$ if the region to the right (resp., left) of the shortest path triangle cycle $(\pi_P(p,r), \pi_P(r,q), \pi_P(q,p))$ is bounded.)

A subproblem is specified by four points, $p_{i'}, p_{j'}, p_i, p_j \in S$, and is indexed by the tuple (i',j',i,j). Here, we assume that $p_{i'} \ne p_{j'}$ and $p_i \ne p_j$, but we allow the possibility that $p_{j'} = p_i$ or that $p_j = p_{i'}$. One can think of the subproblem this way: The objective is to "complete" a geodesically convex cycle, counterclockwise, from p_j to $p_{i'}$, knowing the two "edges" of the geodesically convex hull, $\pi_P(p_{i'},p_{j'})$ and $\pi_P(p_i,p_j)$, which specify the boundary conditions of geodesic convexity. (Path $\pi_P(p_i,p_j)$ specifies the first "edge" and $\pi_P(p_{i'},p_{j'})$ specifies the last "edge" in a geodesically convex curve (dashed blue in Figure 4) from p_i counterclockwise around to $p_{j'}$.) The "done" portion of the problem is the curve going counterclockwise from $p_{i'}$ to $p_{j'}$ (along $\pi_P(p_{i'},p_{j'})$), then along an unspecified geodesically convex curve to p_i, then from p_i to p_j (along $\pi_P(p_i,p_j)$); the subproblem is to complete the cycle in an optimal manner, from p_j to $p_{i'}$, while making certain that the cycle obeys the outer-illumination property. More formally, SubProblem(i',j',i,j) is to find a minimum-length curve, $\mu \subset P$, from p_j to $p_{i'}$ such that μ sees all points of $R_{i'j'ij} = P \cap W_{i'j'i} \cap W_{j'ij}$ that are outside of the region bounded by the cycle $(\pi_P(p_{i'},p_{j'}), \pi_P(p_{j'},p_i), \pi_P(p_i,p_j), \mu)$. Refer to Figure 4.

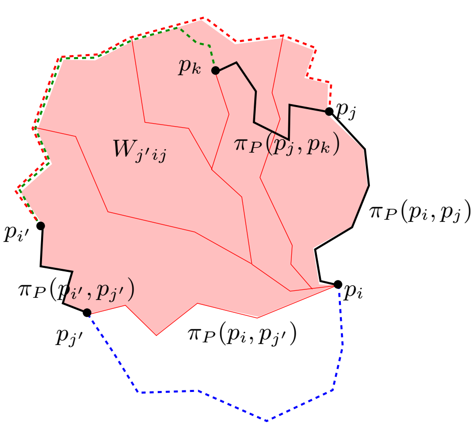

Figure 4: Dynamic programming SubProblem(i',j',i,j) for solving the Discrete-OWRP. The red shaded region depicts the wedge $W_{j'ij}$; the subregion of $W_{j'ij}$ corresponding to the wedge $W_{j'jk}$ is bounded by the path $\pi_P(p_j,p_k)$ and the green dashed path from p_k to $p_{i'}$. (The boundary of P is not shown here, to avoid clutter.)

We say that point $p_k \in S$ is a *valid extension point* for SubProblem(i',j',i,j) if every point of $R_{i'j'ij} \setminus$

$R_{i'j'jk}$ is seen by $\pi_P(p_j, p_k)$; i.e., p_k is a valid extension point if it remains possible, with respect to the outer-illumination requirement for μ, to complete a geodesically convex curve μ from p_j to $p_{i'}$, if the first portion of μ is the shortest path $\pi_P(p_j, p_k)$. We let $S_{i',j',i,j}$ denote the set of all valid extension points for SubProblem(i', j', i, j). We are able to compute these sets in polynomial time using standard methods of computing visibility regions for points and segments in polygonal domains (e.g., using the visibility graph or the visibility complex).

We let $f(i', j', i, j)$ denote the optimal value (minimum length of μ) for SubProblem(i', j', i, j).

Our main recursion, then, is

$$f(i', j', i, j) = \min_{p_k \in S_{i',j',i,j}} \{|\pi_P(p_j, p_k)| + f(i', j', j, k)\},$$

and the base of the recursion is $f(i', j', i, j) = 0$ if $i' = j$ (i.e., $p_j = p_{i'}$, so that a zero-length curve μ closes the cycle by connecting p_j to $p_{i'}$).

Our algorithm tabulates the values $f(i', j', i, j)$ for all subproblems (i', j', i, j) with $i' \neq j', i \neq j$ (but possibly $j' = i$ or $i' = j$), in order of increasing cardinality of the set $S \cap R_{i'j'ij}$, so that the values of the relevant terms $f(i', j', j, k)$ are known before they are needed in solving the recursion for $f(i', j', i, j)$. Our overall objective, then, is to minimize over all $O(n^3)$ choices of $(i', j', i = j', j)$ that are *feasible* the total length, $|\pi_P(p_{i'}, p_{j'})| + |\pi_P(p_{j'=i}, p_j)| + f(i', j', i, j)$, of the closed cycle $(\pi_P(p_{i'}, p_{j'}), \pi_P(p_{j'=i}, p_j), \mu^*_{i'j'ij})$, corresponding to the optimal solution, $\mu_{i'j'ij}$, of SubProblem(i', j', i, j). A triple $(p_{i'}, p_{j'=i}, p_j)$ is *feasible* if there exists an outer-illuminating (geodesically convex) cycle of the form $(\pi_P(p_{i'}, p_{j'}), \pi_{j'=i}, p_j), \mu)$, i.e., if the geodesic convex hull of $(S \cap W_{i'j'j}) \cup \{p_{i'}, p_{j'}, p_j\}$ illuminates all of P outside of the hull.

Correctness follows from Lemma 4.1 and the principle of optimality, since an optimal solution μ for SubProblem(i', j', i, j) must connect p_j to some p_k along a shortest path, with p_k a valid extension point (in order to guarantee visibility coverage outside of the solution curve), and then connect p_k to $p_{i'}$ along a curve that is optimal for SubProblem(i', j', j, k). Our recursion optimizes over all valid choices of p_k. The running time is clearly polynomial, since there are $O(n^4)$ subproblems, and each of the visibility computations can be done efficiently using standard tools. □

As a corollary, we solve an open problem posed in [13]: Is there a polynomial-time algorithm for the watchman route on an arrangement of rays in the plane? With S equal to the vertex set of the arrangement of rays, and considering P to be the union of the rays (an unbounded version of a polygonal domain), we see that the problem is a Discrete-OWRP.

Corollary 4.1. *The WRP on an arrangement of rays in the plane can be solved exactly in polynomial time.*

Further, for the general case of the OWRP, without a given set S constraining the vertex set of a polygonal tour, we obtain a PTAS.

Theorem 4.2. *The OWRP has a PTAS.*

Proof. We know from Lemma 3.3 that there exists a grid-rounded solution to OWRP whose length is at most $(1 + \epsilon)$ times optimal. Thus, we apply our Discrete-OWRP algorithm to the set $S = \mathcal{G}$ of grid points (vertices of $\mathcal{G}^\epsilon_B$), for each choice of MOIS B. □

5 The Inner Watchman Route Problem (IWRP)

The IWRP seeks a minimum-length network that illuminates all of P that is within a given polygonal cycle $\gamma \subset P$. Our approximation result is given in the following theorem:

Theorem 5.1. *Given a polygonal domain P and a simple polygonal cycle $\gamma \subset P$, having in total n vertices, the IWRP has an $O(\log^2 n)$-approximation algorithm.*

Proof. Let $\mathcal{N}^*$ denote an optimal solution for IWRP and let $\mathcal{F}^*$ denote a forest that, together with γ, constitutes a nearly optimal grid-rounded solution for the IWRP, with vertices from the polynomial-size grid $\mathcal{G}$ given by Lemma 3.3. Thus, $\gamma \cup \mathcal{F}^*$ partitions the plane into two connected components, an unbounded component (outside of γ) and a bounded component, Q, which is simply connected. Let Γ^* be the (degenerate) simple polygonal cycle obtained by traversing the boundary of Q; let k denote the number of vertices of Γ^*. (Such a traversal has each edge of $\mathcal{F}^*$ traversed twice, once in each direction.)

Our proof now consists of two parts: (1) We argue that Γ^* can be converted to a "hierarchical geodesic (shortest path) triangulation", consisting of $O(k)$ "inner-illuminating geodesic (shortest path) triangles" covering P_γ and having total perimeter $O(|\Gamma^*| \log k) = O(|\Gamma^*| \log n)$ (since k is polynomial in n). (2) We give an $O(\log n)$-approximation algorithm, based on weighted set cover, for computing a minimum-weight covering of P_γ using "inner-illuminating geodesic (shortest path) triangles", where the weight of a cover is the sum of the perimeters of the geodesic triangles. We argue that the corresponding network of boundaries of geodesic triangles in our covering is a connected network that illuminates all of P_γ. Thus, our solution network has length $O(w^* \log n)$, where w^* is the weight of a minimum-weight covering. Since Γ^* can be converted to a covering of weight $O(|\Gamma^*| \log n)$, we know that $w^* = O(|\Gamma^*| \log n) = O(|\mathcal{N}^*| \log n)$, so our solution has length $O(|\mathcal{N}^*| \log^2 n)$.

For (1), we need to give some definitions. Given a simple polygonal curve $\mu \subset P$, specified by a cyclic sequence of m vertices $(v_1, v_2, \ldots, v_m)$, we will define the "hierarchical geodesic triangulation" of μ with respect to P. For points $u, v, w \in P_\mu \cap P$, the *geodesic triangle*, $\Delta(u, v, w)$, with respect to μ and P is the simple polygonal subset of P_μ bounded by the cycle formed by the three shortest paths $\pi_{P_\mu \cap P}(u, v)$, $\pi_{P_\mu \cap P}(v, w)$, and $\pi_{P_\mu \cap P}(w, u)$, where $\pi_{P_\mu \cap P}(p, q)$ denotes the shortest path from p to q within the polygonal domain $P_\mu \cap P$. (For clarity of exposition, we assume that shortest paths are unique; we can readily account for degenerate cases by defining the notion of a "maximal" geodesic triangle, etc.) We say that a geodesic triangle $\Delta(u, v, w)$ with respect to μ and P is *inner-illuminating* if every point within $\Delta(u, v, w) \cap P$ is seen by some point on the triangle's boundary, $\pi_{P_\mu \cap P}(u, v) \cup \pi_{P_\mu \cap P}(v, w) \cup \pi_{P_\mu \cap P}(w, u)$. The *hierarchical geodesic triangulation* of μ with respect to P is defined to be the decomposition of P_μ into geodesic triangles (with respect to μ and P) given by the geodesic triangles $\Delta(v_1, v_2, v_3)$, $\Delta(v_3, v_4, v_5), \ldots,$ $\Delta(v_1, v_3, v_5)$, $\Delta(v_5, v_7, v_9), \ldots,$ $\Delta(v_1, v_5, v_9)$, $\Delta(v_9, v_{13}, v_{17}), \ldots,$ etc. In other words, the hierarchical geodesic triangulation is given by connecting every 2nd vertex, every 4th vertex, every 8th vertex, etc, of μ with a shortest path (a geodesic "chord") within $P_\mu \cap P$. The geodesic chords comprise a nested set of $O(\log m)$ simple polygons, $\mu_0 = \mu$, μ_1, $\mu_2, \ldots, \mu_{\lceil \log m \rceil - 1}$, with μ_i defined by shortest paths connecting every 2^ith vertex of μ. By the triangle inequality, we see that the length, $|\mu_{i+1}|$, of μ_{i+1} is at most the length, $|\mu_i|$, of μ_i; thus, the total length of all of the boundaries of geodesic triangles in a hierarchical geodesic triangulation is $O(|\mu| \log m)$. Further, if μ is an inner-illuminating curve (i.e., P_μ is inner-illuminating) with respect to P, then every geodesic triangle within μ is also inner-illuminating, since any simple closed within $P_\mu \cap P$ is inner-illuminating (by the Jordan Curve Theorem). To complete part (1) of the argument, we use the hierarchical geodesic triangulation of $\mu = \Gamma^*$. The resulting geodesic triangles are inner-illuminating and cover all of P_γ, and the total length is $O(|\Gamma^*| \log n)$.

For part (2) of the proof, we use the greedy set cover algorithm to solve approximately a weighted set cover instance, which we now describe. Consider all inner-illuminating geodesic triangles within P_γ defined by a triple of grid points of $\mathcal{G}$. Their overlay yields an arrangement of line segments that partitions P_γ into polygonal cells. In fact, since the grid $\mathcal{G}$ includes all vertices of P within P_γ, we can observe that the cells of the arrangement are convex. (Note too that this implies that there exists a covering of P_γ by inner-illuminating geodesic triangles, since a triangulation of $P_\gamma \cap P$ is such a covering – every triangle contained in P is trivially inner-illuminating.) Our weighted set cover instance has elements corresponding to the cells, and sets corresponding to inner-illuminating geodesic triangles within P_γ, with weight equal to the perimeter length. The standard greedy set-cover algorithm [9] yields an $O(\log n)$-approximation to a minimum-weight cover. Finally, we delete any (redundant) geodesic triangle that lies within another geodesic triangle of our cover.

To complete the proof, we argue that the covering by inner-illuminating geodesic triangles given by our algorithm yields a valid solution to the IWRP. First, the network consisting of the union of the boundaries of all geodesic triangles of the covering is connected – if not, since the geodesic triangles fully cover $P_\gamma \cap P$, the only possible disconnection arises if some geodesic triangle lies fully inside another, implying that it is deleted in the final phase of the algorithm. Second, the network illuminates all of $P_\gamma \cap P$, since the inner-illuminating geodesic triangles form a cover $P_\gamma \cap P$, and thus illuminate all of $P_\gamma \cap P$. $\square$

6 The Main Result

Our main result is based on the following algorithm: We enumerate each choice of MOIS, B, and consider the possibility that B is a MOIS within the bounding (square) box of an optimal watchman route, γ^*. This allows us to compute the grid partition $\mathcal{G}_B^\epsilon$, for any fixed $\epsilon > 0$, and know, from the localization Lemma 3.1 and discretization Lemma 3.3, that our search can be restricted to grid-rounded solutions within the enlarged square, $\bar{B}$. We solve the corresponding Discrete-OWRP, yielding a cycle γ, which is then used as the input cycle in the IWRP. Our theorem below proves that this algorithm gives the claimed approximation guarantee.

THEOREM 6.1. *The WRP has an $O(\log^2 n)$-approximation algorithm.*

Proof. Let γ^* be an optimal watchman tour. Its bounding (square) box is outer-illuminating (since it contains γ^*, which must be outer-illuminating), and therefore contains at least one MOIS, B. We know from Lemma 3.1 that γ^* lies within the enlarged box, $\bar{B}$, and we know by Lemma 3.3 that there exists a visibility covering tour, $\gamma_{\mathcal{G}}$, of length at most $(2+\epsilon)|\gamma^*|$, that is polygonal with vertices among the vertices $\mathcal{G}$ of $\mathcal{G}_B^\epsilon$

Since the geodesic convex hull of any closed curve has length at most that of the curve, we know that $|GCH(\gamma_{\mathcal{G}})| \leq |\gamma_{\mathcal{G}}| = O(|\gamma^*|)$. Also, $GCH(\gamma_{\mathcal{G}})$ is polygonal, with vertices among $\mathcal{G}$. Further, since the geodesic convex hull $GCH(\gamma_{\mathcal{G}})$ encloses $\gamma_{\mathcal{G}}$, and $\gamma_{\mathcal{G}}$ sees all of P, we know that the boundary of $GCH(\gamma_{\mathcal{G}})$ sees all of P that is outside of $GCH(\gamma_{\mathcal{G}})$. Since our algorithm iterates over all choices of MOIS, we know that it considered B, and therefore it solved the Discrete-OWRP corresponding to it, and this yielded a tour, γ, whose length obeys $|\gamma| \leq |GCH(\gamma_{\mathcal{G}})| \leq |\gamma_{\mathcal{G}}| = O(\gamma^*)$.

Our algorithm augments γ with a network (specifically, the union of inner-illuminating geodesic triangle boundaries) to give a network $\mathcal{N} \supseteq \gamma$ that sees all of P within γ. By Theorem 5.1, we know that $|\mathcal{N}| \leq C \cdot |IWRP^*_\gamma| \log^2 n$, where $IWRP^*_\gamma$ is an optimal solution to the IWRP for curve γ.

We now claim that γ, together with $\gamma^* \cap P_\gamma$ (the restriction of the optimal solution γ^* to the region surrounded by γ) yields a connected network that illuminates all of P within γ (i.e., is a feasible solution for the IWRP problem specified by γ and P). To see this, note that any point $p \in P_\gamma$ that is not seen by a point of $\gamma^* \cap P_\gamma$ must be seen by a point of γ^* that is outside of γ, implying that it is seen by γ.

Thus, by the claim above, $|IWRP^*_\gamma| \leq |\gamma^* \cap P_\gamma| + |\gamma|$. Since $|\gamma^* \cap P_\gamma| \leq |\gamma^*|$ and $|\gamma| = O(\gamma^*)$, we get that $|\mathcal{N}| \leq O(|\gamma^*| \log^2 n)$. □

7 Hardness of Approximation

THEOREM 7.1. *The watchman route problem in a planar polygonal domain cannot be approximated in polynomial time within a factor $c \log n$, for some constant $c > 0$, assuming $P \neq NP$.*

Proof. Our reduction is from SET-COVER, which is known (see, e.g., [1]) not to have a polynomial-time approximation algorithm with factor $c \log n$, for some constant $c > 0$, assuming $P \neq NP$. Consider an instance of SET-COVER, with universe set $U = \{x_1, x_2, \ldots, x_n\}$, and collection of subsets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, with $S_j \subseteq U$.

Our construction is based on a polygonal domain P shown in Figure 5 (not to scale). The black lines in the figure on the right are narrow corridors; we scale the construction so that the corridors are width 1. We use hollow circles to represent the m sets S_j, which lie on a horizontal row at height L above the base corridor (the "backbone" of a "comb" whose "teeth" correspond to the sets S_j), of width $O(L)$. The n elements x_i are represented by hollow circles closely spaced along a horizontal line at height $L + 3mL$ above the base. (The circles are not part of the construction – corridors remain straight at their crossings.) Corridors link S_j with each element $x_i \in S_j$. In the right figure, it is not possible to discern the connections, so we spread out the x_i's in the middle corridor. Here, the distance of $3mL$ from sets to elements has been distorted – we are depicting the fact that all of the crossings among the corridors are to take place at a height of (about) $3mL$ above the sets. In addition, in the middle figure we highlight in red one additional corridor, the "back hallway", extending from each x_i to the base. Finally, the left figure zooms in on vicinity of an element x_i, showing the presence of a line segment obstacle/hole, ab. The shaded region, R_i, is the locus of points with the corridors incident on x_i that see a point on the *right* side of ab, very close to a; all of the left side of ab is seen along the back hallway associated with x_i.

In order for a watchman to see the base corridor, he must visit the base. In order to visit each of the visibility polygons R_i, a watchman must venture up to the S_j level of at least one set S_j that contains x_i. While corridors cross, they do so at positions that are very far above the S_j's – so far that it is surely not worth it to travel there (as a watchman can traverse the base and go up each of the m corridors to every single set S_j at a cost of only about $O(L) + 2mL$). Thus, an approximating watchman route must traverse the base corridor, and *selectively* venture up to the S_j level in other corridors, in order to yield a covering of the x_i's. The length of the tour is roughly $O(L)+2kL$, if the watchman ventures up to k sets S_j to see all of the domain (in which case the S_j's form a cover). Thus, an approximating watchman tour better than factor $c \log n$ would imply a similar approximation factor for set cover. □

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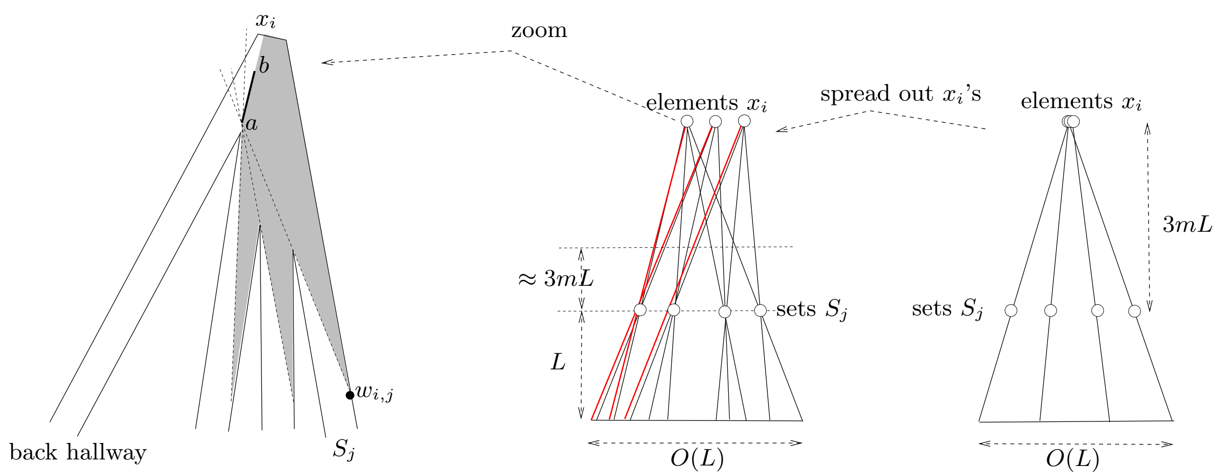

Figure 5: Construction used in proving the reduction from SET-COVER.

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