6.1. Joint Distribution Functions

The joint cumulative probability distribution function of any two random variables $X$ and $Y$ is defined by $F_{X,Y}(a,b) = P\{X \leq a, Y \leq b\}, -\infty < a, b < \infty$.

Note that the joint probability functions are defined in CDF instead of pdf. The reason is that it can accommodates all cases, such as when $X$ and $Y$ are both discrete, both continuous, or one discrete and one continuous, etc.

**Relation between CDF of $X$ and $Y$ with their joint CDF:**

The cumulative distribution function of $X$ is related to the joint cumulative probability distribution function by $F_X(a) = F_{X,Y}(a, \infty)$, and the cumulative distribution function of $Y$ is related to the joint cumulative probability distribution function by $F_Y(b) = F_{X,Y}(\infty, b)$.

In this context, $F_X(a)$ and $F_Y(b)$ are referred to as the marginal distribution functions of $X$ and $Y$ respectively. The probability of a rectangular region is given by

$$P\{a_1 < X \leq a_2, b_1 < Y \leq b_2\} = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F(a_1, b_1).$$

Two discrete random variables:

Definition of joint probability mass function of two discrete random variables: $p(x,y) = P(X = x, Y = y)$. One can calculate the marginal cumulative probability function of $X$ as well as the marginal cumulative probability function of $Y$. Let $X$ and $Y$ have joint probability mass function $p(x,y)$. Let $A$ be the set of possible value of $X$ and $B$ be the set of possible values of $Y$. Then the functions $p_X(x) = \sum_{y \in B} p(x,y)$ and $p_Y(y) = \sum_{x \in A} p(x,y)$ are called, respectively, the marginal (cumulative) distribution functions of $X$ and $Y$.

**Example 1a.** 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. Let $X$ and $Y$ denote, respectively, the number of red and white balls chosen.

Two continuous random variables:

The random variables $X$ and $Y$ are jointly continuous if there exists a function $f(x,y)$ defined for all real $x$ and $y$, having the property that for very set $C$ of pairs of real numbers (that is, $C$ is a set in the two-dimensional plane) $P\{(X,Y) \in C\} = \int \int_{(x,y) \in C} f(x,y)dxdy$. The function $f(x,y)$ is called the joint probability
density function of $X$ and $Y$. The relation between the joint pdf and the joint cdf is

$$f(a,b) = \frac{\partial^2}{\partial a \partial b} F(a,b).$$

Example 1c. The joint density function of $X$ and $Y$ is given by

$$f(x,y) = 2e^{-x}e^{-2y}, 0 < x < \infty, 0 < y < \infty,$$

and zero otherwise. Compute (a) the joint cdf of $X$ and $Y$; (b) $P\{X > 1, Y < 1\}$; (c) $P\{X > 1, Y < 1\}$; (d) $P\{X < Y\}$; and (e) $P\{X < a\}$.

Example 1d. Consider a circle of radius $R$ and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define $X$ and $Y$ to be the coordinates of the point chosen, it follows, since $(X,Y)$ is equally likely to be near each point in the circle, that the joint density function of $X$ and $Y$ is given by

$$f(x,y) = c, \text{ if } x^2 + y^2 \leq R^2,$$

and zero otherwise for some value of $c$.

(a) Determine $c$. (b) Find the marginal density functions of $X$ and $Y$. (c) Compute the probability that $D$, the distance from the origin of the point selected, is less than or equal to $a$. (d) Find $E(D)$. 

End of handout