6.2. Independent Random Variables

The random variables \( X \) and \( Y \) are said to be independent if for any two sets of real numbers \( A \) and \( B \), \( P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\} \). Another way to define random variables \( X \) and \( Y \) to be independent is that the conditional distribution of \( X \) given \( Y = y \) is irrelevant to the value of \( y \), and equals the marginal distribution of \( X \); and the conditional distribution of \( Y \) given \( X = x \) equals the marginal distribution of \( Y \). From the definition of conditional distribution, this is equivalent to \( f_{X|Y=y}(x) = \frac{f(x, y)}{f_Y(y)} = f_X(x) \). Therefore

Their joint cdf is the product of their marginal cdfs \( f(x, y) = f_X(x)f_Y(y) \).

Proposition 2.1: The continuous (discrete) random variables \( X \) and \( Y \) are independent if and only if their joint probability density (mass) function can be expressed as \( f_{X,Y}(x, y) = h(x)g(y), -\infty < x < \infty, -\infty < y < \infty \).

If two random variables \( X \) and \( Y \) are independent, functions of \( X \) are independent of functions of \( Y \).

If two random variables \( X \) and \( Y \) are independent, expectations of the product of a function of \( X \) and of a function of \( Y \) are the product of the expectations.

Example 2f. If the joint density function of \( X \) and \( Y \) is \( f_{X,Y}(x, y) = 24xy, 0 < x < 1, 0 < y < 1, 0 < x + y < 1 \), and zero otherwise, are \( X \) and \( Y \) independent?

Example 2h. Let \( X, Y, \) and \( Z \) be independent and uniformly distributed over \( 0,1 \). Compute \( P\{X < YZ\} \).

6.3. Sums of Independent Random Variables

In a gambling problem, let \( X \) be the winnings from one play of a game of chance and have pdf \( f_X \), and \( Y \) be the winnings from one play of another game of chance with pdf \( f_Y \), with \( X \) and \( Y \) independent. The random variable \( S = X + Y \) represents the total winnings from the two games.

Then \( F_S(s) = F_{X+Y}(s) = P\{X + Y \leq s\} = \int_{-\infty}^{s} F_X(s-y)f_Y(y)dy \). Further, \( f_{X+Y}(s) = \int_{-\infty}^{s} f_X(s-y)f_Y(y)dy \).

Example 3a. Sum of two independent uniform random variables. If \( X \) and \( Y \) are two independent random variables, both uniformly distributed on \((0,1)\) calculate the probability density of \( S = X + Y \).
**Example 3e. Sums of independent Poisson random variables.** If $X$ and $Y$ are independent Poisson random variables with respective parameters $\lambda_1$ and $\lambda_2$, compute the distribution of $X + Y$.

Proposition 3.1.
If $X$ and $Y$ are independent gamma random variables with respective parameters $(s, \lambda)$ and $(t, \lambda)$, then $S = X + Y$ is a gamma distribution with parameters $(s + t, \lambda)$.

If $X_i, i = 1, \ldots, n$ are independent gamma random variables with respective parameters $(t_i, \lambda)$, then $S_n = \sum_{i=1}^{n} X_i$ is gamma with parameters $(\sum_{i=1}^{n} t_i, \lambda)$.

Special cases of Gamma distribution

- **Gamma(1,2/2):** Chi-Squared distribution with $n$ degrees of freedoms
- **Gamma(1,\lambda):** Exponential distribution with parameter $\lambda$

Example 3b. Let $X_1, X_2, \ldots, X_n$ be $n$ independent exponential random variable, then $S = X_1 + X_2 + \ldots + X_n$ is a gamma random variable with parameters $(n, \lambda)$

Proposition 3.2.
If $X_i, i = 1, \ldots, n$ are independent normally distributed random variables with respective parameters $(\mu_i, \sigma_i^2)$, then $S_n = \sum_{i=1}^{n} X_i$ is normal with expected value $\sum_{i=1}^{n} \mu_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$.

End of handout