AMS 311
Spring Semester, 2004

Chapter Eight: Limit Theorems

8.2 Chebyshev's Inequality and the Weak Law of Large Numbers

Proposition 2.1 ***Markov's inequality***

If X is a random variable that takes only nonnegative values, then for any value a>0,

$$P\{X \geq a\} \leq \frac{E[X]}{a}.$$

Proposition 2.2 ***Chebyshev's inequality***

If X is a random variable with finite mean μ and variance σ^2, then for any value $k > 0$,

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$

Example 2a. Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

a) What can be said about the probability that this week's production will exceed 75?
b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 and 60?

Proposition 2.3
If $\text{var}(X) = 0$, then $P\{X = E[X]\} = 1$.

Theorem 2.1 ***The weak law of large numbers***
Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables each having finite mean $E[X_i] = \mu$. Then, for any

$\varepsilon > 0$, $P\{|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon\} \to 0$ as $n \to \infty$.

8.3 Central Limit Theorem

Central Limit Theorems

The most basic applications are concerned with sums and averages of random samples. Remember that a random sample of size n from the random variable X is defined to be a set of n independently and identically distributed random variables, each with the same marginal distribution as X.

As you learned in AMS 310, the two basic random variables we are concerned with are

$S_n = \sum_{i=1}^{n} X_i$ and $\overline{X}_n = \frac{S_n}{n} = \frac{\sum_{i=1}^{n} X_i}{n}$.

Basic moment calculations show that

$$E[S_n] = \sum_{i=1}^{n} E[X_i] = nE[X] \text{ and } E[\overline{X}_n] = \frac{E[S_n]}{n} = \frac{E[\sum_{i=1}^{n} X_i]}{n} = E[X].$$

Standard variance calculations show that

$$\operatorname{var}(S_n) = \operatorname{var}(\sum_{i=1}^{n} E[X_i]) = n\operatorname{var}(X) \text{ and } \operatorname{var}(\overline{X}_n) = \frac{\operatorname{var}(S_n)}{n^2} = \frac{\operatorname{var}(X)}{n}.$$

The central limit theorem adds the fact that the distribution of these random variables becomes closer to normal as the number in the random sample increases.

Modern proofs use a function called the moment generating function. For those who know complex analysis, there is a generalization of the moment generating function called the characteristic function that is preferred (because it always exists).

Theorem 3.1 The central limit theorem

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. Then the distribution of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is, for $-\infty < a < \infty$,

$$P\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx \text{ as } n \to \infty.$$

Example central limit theorem problem:

The winnings W in a game of chance have an expected value \$25 and variance 1,000,000. Describe the distribution of $S_{100} = \sum_{i=1}^{100} W_i$, the total winnings after 100 independent plays of the game of chance. What is $P(S_{100} \leq 0)$? What is the value of the reserve r that you should hold so that $P(S_{100} \leq r) = 0.01$? How many times n would a gambler have to play this game of chance so the probability that S_n is greater than 0 is 0.99?

End of handout