PROBABILITY THEORY

Answer to exam 2

Problem 1

a) Finding q

From the CDF of X we have:

\[ P(X = -2) = 0.2, \quad P(X = 0) = q - 0.2, \quad P(X = 1) = 1 - q. \]

Using the fact that \( E(X) = 0 \), we can solve for q. By definition,

\[ E(X) = \sum_{\text{all } x} x P(X = x) \]

This is equivalent to

\[ -2 \cdot 0.2 + 0 \cdot (q - 0.2) + 1 \cdot (1 - q) = 0 \Rightarrow -0.4 + 1 - q = 0. \]

Therefore, \( q = 0.6 \).

b) Bar graphs with X on the X-axis and P(X=x) on the Y-axis.

c) \( P(X^2 \geq 2 \mid X \geq 0) = 1 - P(X^2 \leq 2 \mid X \geq 0) \)

\[ = 1 - \frac{P(\sqrt{2} \leq X \leq \sqrt{2} \mid X \geq 0)}{P(X \geq 0)} \]

\[ = 1 - \frac{P(X=0) + P(X=1)}{P(X=0) + P(X=1)} = 1 - 1 = 0 \]

\[ E\left(\frac{1}{X+1}\right) = \sum_{\text{all } x} \frac{1}{x+1} P(X = x) \]

\[ E\left(\frac{1}{X+1}\right) = \frac{1}{2+1} \cdot 0.2 + \frac{1}{0+1} \cdot 0.4 + \frac{1}{1+1} \cdot 0.4 \]

\[ = -0.2 + 0.4 + 0.2 = 0.4 \]

Problem 2

a) Finding c

c is such \( \int_{c}^{\infty} \frac{1}{x^2} \, dx = 1 \Rightarrow \frac{c}{100} = 1 \Rightarrow c = 100. \)

b) CDF of Y = 2X + 10

\[ F_{y}(Y) = P(Y < y) = P(2X + 10 < y) = P(2X < y - 10) = P(X < \frac{y-10}{2}) \]

\[ = F_{x}(\frac{y-10}{2}) = \int_{100}^{\frac{y-10}{2}} \frac{100}{x^2} \, dx = 100 \left( \frac{1}{100} - \frac{2}{y-10} \right) = \frac{y-210}{y-10} = 1 - \frac{200}{y-10} \]

Therefore, the CDF of Y is written as follows:

\[ F_{y}(y) = \begin{cases} 
0 & \text{if } y < 210 \\
1 - \frac{200}{y-10} & \text{if } 210 \leq y < \infty 
\end{cases} \]

c) Taking derivative of \( F_{y}(y) \) with respect to y, we have:
\[ f_y(y) = \begin{cases} \frac{200}{(y - 10)^2} & 210 \leq y < \infty \\ 0 & \text{Otherwise} \end{cases} \]

**Problem 3**

Let \( X \) represent the number of car accidents that happen today. \( X \) follows a Poisson (\( \frac{3}{7} \)).

\[
P(X \geq 2) = 1 - P(X = 1) = 1 - (P(X = 0) + P(X = 1)) = 1 - e^{\frac{3}{7}} - 3e^{\frac{3}{7}} = 1 - 4e^{\frac{3}{7}}.
\]

**Problem 4**

Let \( X \) represent the time (in hours) that it takes to repair a machine \( X \) follows an \( \text{Exp}(1) \)

a) \( P(X > 2) = \int_2^\infty e^{-x} \, dx = e^{-2} \)

b) \( P(X < \frac{5}{4} | X > 4) = 1 - P(X > \frac{5}{4} | X > 4) = 1 - P(X > 1) = 1 - \int_1^\infty e^{-x} \, dx = 1 - e^{-1} \)

**Bonus**

Distribution of \(-\log(X)\)

\[
F_y(Y) = P(Y < y) = P(-\log(X) < y) = P(\log(X) > -y) = P(X > e^{-y}) = 1 - F_x(e^{-y}) = 1 - \int_0^{e^{-y}} dx = 1 - e^{-y}.
\]

Which is the CDF of an exponential distribution with \( \lambda = 1 \).