AMS 311 (Fall, 2004) Jasmin Divers

PROBABILITY THEORY

Answer to homework 8

Problem 1

$$f(x,y) = \begin{cases} \frac{1}{x^2y^2} & x \geq 1, y \geq 1 \\ 0 & Otherwise \end{cases}$$

1. Let $U = XY$ and $V = \frac{X}{Y}$. Solving the following for system for X and Y leads to $X = \sqrt{UV}$ and $Y = \sqrt{\frac{U}{V}}$

a) *Joint distribution of U and V*

The Jacobian of this transformation is obtained by taking derivatives of U and V with respect to X and Y.

We have:

$\frac{\partial U}{\partial X} = Y$, $\frac{\partial U}{\partial Y} = X$, $\frac{\partial V}{\partial X} = \frac{1}{Y}$, $\frac{\partial U}{\partial Y} = -\frac{X}{Y^2}$

Therefore, we have

$$J = \begin{vmatrix} Y & x \\ \frac{1}{Y} & -\frac{X}{Y^2} \end{vmatrix} = -2\frac{X}{Y} = -2V$$

By definition, we have:

$f_{U,V}(u,v) = |J|^{-1} f(X(U,V), Y(U,V))$

Which leads to

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{2u^2v} & u \geq 1, \frac{1}{u} < v < u \\ 0 & Otherwise \end{cases}$$

$$f_U(u) = \begin{cases} \int_{\frac{1}{u}}^{u} \frac{1}{2u^2v} du = \frac{1}{u^2} logu & u \geq 1 \\ 0 & Otherwise \end{cases}$$

$$f_V(v) = \begin{cases} \int_{\frac{1}{v}}^{\infty} \frac{1}{2u^2v} du = \frac{1}{2v^2} & v > 1 \\ \int_{\frac{1}{2}}^{\infty} \frac{1}{2u^2v} du = \frac{1}{2} & 0 < v < 1 \\ 0 & Otherwise \end{cases}$$

2. Let $u = x + y$, $v = x + z$ and $w = y + z$. Taking derivative of x,y, and z with respect to u,v and w leads to the following Jacobian value:

$$|J| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = |-2| = 2$$

Solving for x, y and z leads to $x = \frac{u+v-w}{2}$, $y = \frac{u+v-w}{2}$, and $z = \frac{-u+v+w}{2}$
The joint distribution of u,v and w is given by:
$f(u,v,w) = \frac{1}{2}e^{-\frac{1}{2}(u+v+w)}$ for $u+v > w$, $u+w > v$ and $u+v+w > 0$.

Problem 2
At $Y = y, X|Y \sim U(0, y+1)$.

Therefore, $f_{X|Y=y}(x,y) = \begin{cases} \frac{1}{y+1} & 0 < x < y+1 \\ 0 & Otherwise \end{cases}$

Since it is known that $f_Y(y) = y$ for $0 < y < \sqrt{2}$, the joint density of X and Y is the product of the conditional distribution of X given Y and the marginal distribution of Y.
a) Joint density of X and Y
Given that Y=y,

$$f_{X,Y}(x,y) = \begin{cases} \frac{y}{y+1} & 0 < x < y+1, 0 < y < \sqrt{2} \\ 0 & Otherwise \end{cases}$$

b) Marginal density of X

$$f_X(x) = \begin{cases} \int_0^{\sqrt{2}} \frac{y}{y+1}dy = \sqrt{2} - \ln(\sqrt{2}+1) & 0 < x < 1 \\ \int_{x-1}^{\sqrt{2}} \frac{y}{y+1}dy = \ln x - x + \sqrt{2} + 1 - \ln(\sqrt{2}+1) & 1 < x < \sqrt{2}+1 \\ 0 & Otherwise \end{cases}$$

For more information look at the graph. **Problem 3**
$E[(X-Y)^2] = E(X^2 - 2XY + Y^2) = E(X^2) - 2E(X)E(Y) + E(Y^2)$ since X and Y are independent.
Now using the fact that both X and Y are uniformly distributed over (0,1), we have: $E(X^2) = E(Y^2) = \frac{1}{3}$ and $E(X) = E(Y) = \frac{1}{2}$.
Therefore, $E[(X-Y)^2] = 2\times \frac{1}{3} - 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{6}$

Problem 4
Problem 37
Define X as the value of the first roll and Y the value of second roll. Let S=X+Y be the sum X and Y. Also, let D be their difference.
That is: D=X-Y.
$Cov(S,D) = Cov(X+Y, X-Y) = Cov(X, X-Y) + Cov(Y, X-Y) = Var(X) + Cov(X,Y) - Cov(X,Y) - Var(Y) = Var(X) - Var(Y) = 0$. Since this is the same experiment conducted twice. The variance is the same at each trial.

Problem 38

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$E(XY) = \int_0^\infty \int_0^x y 2e^{-2x} dy dx = \frac{1}{4}$

$E(X) = \int_0^\infty x(\int_0^x \frac{2e^{-2x}}{x} dy) dx = \int_0^\infty 2e^{-2x} dx = \frac{1}{2}$

$E(Y) = \int_0^\infty y(\int_y^\infty \frac{2e^{-2x}}{x} dx) dy = \int_0^\infty y^2 e^{-2y} dy = \frac{1}{4}$

$Cov(X,Y) = \frac{1}{4} - \frac{1}{4}\frac{1}{2} = \frac{1}{8}$

Problem 5

a) $M_x(t) = E(e^{tx}) = \frac{1}{3}(e^{-t} + e^{3t} + e^{7t})$

$E(X) = M'_x(0) = \frac{1}{3}(-1 + 3 + 7) = \frac{9}{3} = 3$

$E(X^2) = M''_x(0) = \frac{1}{3}(1 + 9 + 49) = \frac{59}{3}$

$Var(X) = \frac{59}{3} - 3^2 = \frac{29}{3}$.

Computing directly, we have $E(X) = \frac{1}{3}(-1 + 3 + 7) = \frac{9}{3} = 3$

$E(X^2) = \frac{1}{3}((-1)^2 + 3^2 + 7^2) = \frac{1}{3}(1 + 9 + 49) = \frac{59}{3}$

b) $M_x(t) = e^{2(e^t - 1)} \Rightarrow X \sim Poisson(2)$

$M_x(t) = (\frac{3}{4}e^t + (1 - \frac{3}{4}))^1 \Rightarrow Y \sim Bin(1, \frac{3}{4})$

$P(XY = 0) = P(X = x, Y = 0) + P(X = 0, Y = y)$

$\quad = P(X = x)P(Y = 0) + P(X = 0)P(Y = y)\ \forall X > 0, y = 0, 1.$

Therefore,$P(X = x) = 1$ and $P(Y = y) = 1$.

$P(XY = 0) = P(X = 0) + P(Y = 0) = e^{-2} + \frac{1}{4}$

$E(XY) = E(X)E(Y) = 2\frac{3}{4} = \frac{3}{2}$