48. (a) $E[X] = 6$

(b) $E[X \mid Y = 1] = 1 + 6 = 7$

(c) $1\frac{1}{5} + 2\frac{4}{5}\frac{1}{5} + 3\left(\frac{4}{5}\right)^2\frac{1}{5} + 4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^4(5+6)$

49. $$E[X \mid Y = i] = \begin{cases} 3 \cdot \frac{3}{5} = \frac{9}{5}, & i = 1 \\ 3 \cdot \frac{2}{5} = \frac{6}{5}, & i = 2 \\ 3 \cdot \frac{1}{5} = \frac{3}{5}, & i = 3 \\ 0, & i = 4 \end{cases}$$

50. $$f_{X|Y}(x \mid y) = \frac{e^{-x/y}e^{-y}/y}{\int_0^\infty e^{-x/y}e^{-y}/y\,dx} = \frac{1}{y}e^{-x/y}, \quad 0 < x < \infty$$

Hence, given $Y = y$, X is exponential with mean y, and so

$$E[X^2 \mid Y = y] = 2y^2$$

P-2

51. $$f_{X|Y}(x \mid y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y\,dx} = \frac{1}{y}, \quad 0 < x < y$$

$$E[X^3 \mid Y = y] = \int_0^y x^3\frac{1}{y}dx = y^3/4$$

52. The average weight, call it $E[W]$, of a randomly chosen person is equal to average weight of all the members of the population. Conditioning on the subgroup of that person gives

$$E[W] = \sum_{i=1}^r E\{W \mid \text{member of subgroup } i]p_i = \sum_{i=1}^r w_i p_i$$

53. Let X denote the number of days until the prisoner is free, and let I denote the initial door chosen. Then

$$E[X] = E[X \mid I = 1](.5) + E[X \mid I = 2](.3) + E[X \mid I = 3](.2)$$
$$= (2 + E[X])(.5) + (4 + E[X])(.3) + .2$$

Therefore,

$$E[X] = 12$$

Chapter 8

→ P-3

Problems

1. $P\{0 \le X \le 40\} = 1 - P\{|X - 20| > 20\} \ge 1 - 20/400 = 19/20$

2. (a) $P\{X \ge 85\} \le E[X]/85 = 15/17$

   (b) $P\{65 \le X \le 85) = 1 - P\{|X - 75| > 10\} \ge 1 - 25/100$

   (c) $P\left\{\left|\sum_{i=1}^{n} X_i/n - 75\right| > 5\right\} \le \dfrac{25}{25n}$ so need $n = 10$

3. Let Z be a standard normal random variable. Then,

$$P\left\{\left|\sum_{i=1}^{n} X_i/n - 75\right| > 5\right\} \approx P\{|Z| > \sqrt{n}\} \le .1 \text{ when } n = 3$$

→ P-4

4. (a) $P\left\{\sum_{i=1}^{20} X_i > 15\right\} \le 20/15$

   (b) $$P\left\{\sum_{i=1}^{20} X_i > 15\right\} = P\left\{\sum_{i=1}^{20} X_i > 15.5\right\}$$
$$\approx P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\}$$
$$= P\{Z > -1.006\}$$
$$\approx .8428$$

5. Letting X_i denote the i^{th} roundoff error it follows that $E\left[\sum_{i=1}^{50} X_i\right] = 0$,

$\text{Var}\left(\sum_{i=1}^{50} X_i\right) = 50\ \text{Var}(X_1) = 50/12$, where the last equality uses that $.5 + X$ is uniform (0, 1) and so $\text{Var}(X) = \text{Var}(.5 + X) = 1/12$. Hence,

$$P\left\{\left|\sum X_i\right| > 3\right\} \approx P\{|N(0, 1)| > 3(12/50)^{1/2}\} \text{ by the central limit theorem}$$
$$= 2P\{N(0, 1) > 1.47 = .1416$$

6. If X_i is the outcome of the i^{th} roll then $E[X_i] = 7/2$ $\text{Var}(X_i) = 35/12$ and so

$$P\left\{\sum_{i=1}^{79} X_i \le 300\right\} = P\left\{\sum_{i=1}^{79} X_i \le 300.5\right\}$$
$$\approx P\left\{N(0,1) \le \frac{300.5 - 79(7/2)}{(79 \times 35/12)^{1/2}}\right\} = P\{N(0,1) \le 1.58\} = .9429$$

Bonus - 1

55. Let N denote the number of ducks. Given $N = n$, let $I_1, \ldots, I_n$ be such that

$$I_i = \begin{cases} 1 & \text{if duck } i \text{ is hit} \\ 0 & \text{otherwise} \end{cases}$$

$$E[\text{Number hit} \mid N = n] = E\left[\sum_{i=1}^{n} I_i\right]$$
$$= \sum_{i=1}^{n} E[I_i] = n\left[1 - \left(1 - \frac{.6}{n}\right)^{10}\right], \text{ since given}$$

$N = n$, each hunter will independently hit duck i with probability $.6/n$.

$$E[\text{Number hit}] = \sum_{n=0}^{\infty} n\left(1 - \frac{.6}{n}\right)^{10} e^{-6} 6^n / n!$$

Bonus - 2

56. Let $I_i = \begin{cases} 1 & \text{elevator stops at floor } i \\ 0 & \text{otherwise} \end{cases}$. Let X be the number that enter on the ground floor.

$$E\left[\sum_{i=1}^{N} I_i \middle| X = k\right] = \sum_{i=1}^{N} E[I_i | X = k] = N\left[1 - \left(\frac{N-1}{N}\right)^k\right]$$

$$E\left[\sum_{i=1}^{N} I_i\right] = N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k e^{-10}\frac{(10)^k}{k!}$$
$$= N - Ne^{-10/N} = N(1 - e^{-10/N})$$

57. $E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X] = 12.5$

58. Let X denote the number of flips required. Condition on the outcome of the first flip to obtain.

$$E[X] = E[X \mid \text{heads}]p + E[x \mid \text{tails}](1-p)$$
$$= [1 + 1/(1-p)]p + [1 + 1/p](1-p)$$
$$= 1 + p/(1-p) + (1-p)/p$$

59. Let X be the number of flips required and use the hint.

$$E[X] = \sum_{i=1}^{3} E[X|T = i]p^{i-1}(1-p) + E[X|T = 0]p^3$$
$$= \sum_{i=1}^{3}(i + E[X])p^{i-1}(1-p) + 3p^3$$
$$= \sum_{i=1}^{3} ip^{i-1}(1-p) + E[X]\sum_{i=1}^{3} p^{i-1}(1-p) + 3p^3$$

Now solve for $E[X]$.