

# PROBABILITY THEORY

## Answer to the practice Exam

### Problem 1

Let  $X$  be the number of spades. One has to remember that a deck of cards has 13 spades.

$X$  can take one of the following values: 0, 1, 2, and 3.

$$P(X = 0) = \frac{\binom{13}{0} \times \binom{39}{3}}{\binom{52}{3}}$$

$$P(X = 1) = \frac{\binom{13}{1} \times \binom{39}{2}}{\binom{52}{3}}$$

$$P(X = 2) = \frac{\binom{13}{2} \times \binom{39}{1}}{\binom{52}{3}}$$

$$P(X = 3) = \frac{\binom{13}{3} \times \binom{39}{0}}{\binom{52}{3}}$$

### Problem 2

a) Finding  $q$

First one has to realize that  $q = 0.2 + t$  where  $t$  is the probability that  $X = 0$ . Now,  $P(X^2 > 0) = 0.8 \Rightarrow 1 - P(X^2 = 0) = 0.8 \Rightarrow P(X^2 = 0) = 0.2 \Leftrightarrow P(X = 0) = 0.2$ . Therefore  $q = 0.2 + 0.2 = 0.4$ .

b) The probability mass function of  $X$  can be written as follows:

$$P(X) = \begin{cases} 0.2 & x = -1 \\ 0.2 & x=0 \\ 0.2 & x=1 \\ 0.1 & x=2 \\ 0.3 & x=4 \\ 0 & \text{Otherwise} \end{cases}$$

c) The plot should be a line graph with the values of  $X$  on the abscissa and  $P(X)$  on the ordinate.

$$d) P(X^2 - 1 \geq 3) = P(X^2 \geq 4) = 1 - P(X^2 < 4) = 1 - P(-2 < X < 2)$$

$$= 1 - [P(X = -1) + P(X = 0) + P(X = 1)] = 0.4$$

$$\begin{aligned} \text{e) } P(2X - 3 \geq 2 | X \geq 0.1) &= P(X \geq 2.5 | X \geq 0.1) = \frac{P[(X \geq 2.5) \cap (X \geq 0.1)]}{P(X \geq 0.1)} \\ &= \frac{P(0.1 \leq X \leq 2.5)}{P(X \geq 0.1)} = \frac{P(X=1) + P(X=2)}{P(X=1) + P(X=2) + P(X=4)} = \frac{0.1 + 0.3}{0.1 + 0.3 + 0.3} \\ &= \frac{4}{7} \end{aligned}$$

$$\text{f) } E(X) = \sum_{\text{all } x} xP(X = x) = -1 \times 0.2 + 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.1 + 4 \times 0.3 = 1.4$$

$$\text{g) } E(P(X)) = \sum_{\text{all } x} P(X = x) \times P(X = x) = 3 \times 0.2^2 + 0.1^2 + 0.3^2 = 0.22$$

### Problem 3

$$\text{a) } \text{Var}(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{20}^{60} x^2 \left(\frac{x-20}{800}\right) dx = \frac{1}{800} \left( \int_{20}^{60} x^3 dx - 20 \int_{20}^{60} x^2 dx \right) = \frac{6800}{3}$$

$$E(X) = \int_{20}^{60} x \left(\frac{x-20}{800}\right) dx = \frac{1}{800} \left( \int_{20}^{60} x^2 dx - 20 \int_{20}^{60} x dx \right) = \frac{140}{3}$$

$$\text{Var}(X) = \frac{6800}{3} - \left(\frac{140}{3}\right)^2 = \frac{800}{9}$$

$$\text{b) } P(20 \leq X \leq 40) = \int_{20}^{40} \frac{x-20}{800} dx = \frac{1}{4}$$

$$\text{c) } F_y(y) = P(Y < y) = P(2X < y) = P(X < \frac{y}{2}) = F_x\left(\frac{y}{2}\right)$$

Taking derivative of both sides with respect to y leads to:

$$f_y(y) = \frac{1}{2} f_x\left(\frac{y}{2}\right) \Rightarrow f_y(y) = \begin{cases} 0 & y < 40 \\ \frac{y-40}{3200} & 40 \leq y \leq 120 \\ 1 & y \geq 1 \end{cases}$$

d)

### Problem 4

Let X denote the number of hurricanes per year; X follows a Poisson(5.2).

$$P(X \leq 3) = \sum_{i=0}^3 \frac{e^{-5.2} \times (5.2)^i}{i!}$$

### Problem 5

a) Let X denote the lifetime a radio. It assumed that X follows an Exp(5)  
 $P(X < 5 | X > 2) = 1 - P(X > 5 | X > 2)$ . Using the memoryless property of the exponential distribution, we have  $P(X < 5 | X > 2) = 1 - P(X > 5) = 1 - e^{-25}$ .

b) Let's find the probability that at a radio lasts more than 15 years.

$$P(X > 15) = \int_{15}^{\infty} 5e^{-5x} dx = e^{-75}.$$

Let  $Y$  be the number of radios that last more than 15 years.  $Y$  follows a *Binomial*(20,  $e^{-75}$ ).

We need to compute  $P(Y \geq 1)$ .

$$P(Y \geq 1) = 1 - P(Y = 0)$$

$$P(Y = 0) = \binom{20}{0} (e^{-75})^0 (1 - e^{-75})^{20} \simeq 1. \text{ Therefore, } P(Y \geq 1) \simeq 0.$$