AMS 311 (Fall, 2004) Jasmin Divers

PROBABILITY THEORY

Answer to the practice Exam

Problem 1

Let X be the number of spades. One has to remember that a deck of cards has 13 spades.
X can take one of the following values: 0, 1, 2, and 3.

$P(X = 0) = \frac{\binom{13}{0} \times \binom{39}{3}}{\binom{52}{3}}$

$P(X = 1) = \frac{\binom{13}{1} \times \binom{39}{2}}{\binom{52}{3}}$

$P(X = 2) = \frac{\binom{13}{2} \times \binom{39}{1}}{\binom{52}{3}}$

$P(X = 3) = \frac{\binom{13}{3} \times \binom{39}{0}}{\binom{52}{3}}$

Problem 2

a) Finding q

First one has to realize that $q = 0.2 + t$ where t is the probability that $X = 0$. Now, $P(X^2 > 0) = 0.8 \Rightarrow 1 - P(X^2 = 0) = 0.8 \Rightarrow P(X^2 = 0) = 0.2 \Leftrightarrow P(X = 0) = 0.2$. Therefore $q = 0.2 + 0.2 = 0.4$.

b) The probability mass function of X can be written as follows:

$$P(X) = \begin{cases} 0.2 & x = -1 \\ 0.2 & \text{x=0} \\ 0.2 & \text{x=1} \\ 0.1 & \text{x=2} \\ 0.3 & \text{x=4} \\ 0 & Otherwise \end{cases}$$

c)The plot should be a line graph with the values of X on the abscissa and P(X) on the ordinate.

d)$P(X^2 - 1 \geq 3) = P(X^2 \geq 4) = 1 - P(X^2 < 4) = 1 - P(-2 < X < 2)$

$= 1 - [P(X=-1) + P(X=0) + P(X=1)] = 0.4$

e) $P(2X-3 \geq 2|X \geq 0.1) = P(X \geq 2.5|X \geq 0.1) = \frac{P[(X\geq 2.5) \bigcap (X \geq 0.1)]}{P(X \geq 0.1)}$
$= \frac{P(X\geq 2.5)}{P(X\geq 0.1)} = \frac{P(X=4)}{P(X=1)+P(X=2)+P(X=4)} = \frac{0.3}{0.1+0.3+0.3}$
$= \frac{3}{7}$

f) $E(X) = \sum_{allx} xP(X=x) = -1\times 0.2 + 0\times 0.2 + 1\times 0.2 + 2\times 0.1 + 4\times 0.3 = 1.4$

g) $E(P(X)) = \sum_{allx} P(X=x) \times P(X=x) = 3\times\ 0.2^2 + 0.1^2 + 0.3^2 = 0.22$

Problem 3

a)$Var(X) = E(X^2) - (E(X))^2$
$E(X^2) = \int_{20}^{60} x^2(\frac{x-20}{800})dx = \frac{1}{800}(\int_{20}^{60} x^3 dx - 20\int_{20}^{60} x^2 dx) = \frac{6800}{3}$
$E(X) = \int_0^{60} x(\frac{x-20}{800})dx = \frac{1}{800}(\int_{20}^{60} x^2 dx - 20\int_{20}^{60} x dx) = \frac{140}{3}$
$Var(X) = \frac{6800}{3} - (\frac{140}{3})^2 = \frac{800}{9}$

b) $P(20 \leq X \leq 40) = \int_{20}^{40} \frac{x-20}{800} dx = \frac{1}{4}$

c) $F_y(y) = P(Y < y) = P(2X < y) = P(X < \frac{y}{2}) = F_x(\frac{y}{2})$

$$F_y(y) = \begin{cases} 0 & y < 40 \\ \int_{20}^{\frac{y}{2}} \frac{x-20}{800} dx = \frac{y^2-80y}{6400} & 40 \leq y \leq 120 \\ 1 & y \geq 120 \end{cases}$$

d)

Problem 4

Let X denote the number of hurricanes per year; X follows a Poisson(5.2).
$P(X \leq 3) = \sum_{i=0}^{3} \frac{e^{-5.2}\times(5.2)^i}{i!}$

Problem 5

a) Let X denote the lifetime a radio. It assumed that X follows an Exp(5) $P(X < 5|X > 2) = 1 - P(X > 5|X > 2)$. Using the memoryless property of the exponential distribution, we have $P(X < 5|X > 2) = 1 - P(X > 3 + 2|X > 2) = 1 - P(X > 3) = 1 - e^{-15}$.

b) Let's find the probability that at a radio lasts more than 15 years.
$P(X > 15) = \int_{15}^{\infty} 5e^{-5x} dx = e^{-75}$.

Let Y be the number of radios that last more than 15 years. Y follows a $Binomial(20, e^{-75})$.
We need to compute $P(Y \geq 1)$.
$P(Y \geq 1) = 1 - P(Y = 0)$
$P(Y = 0) = \binom{20}{0}(e^{-75})^0(1 - e^{-75})^{20} \simeq 1$. Therefore, $P(Y \geq 1) \simeq 0$.