PROBABILITY THEORY

Answer to the practice Exam

Problem 1
Let \( X \) be the number of spades. One has to remember that a deck of cards has 13 spades.

\( X \) can take one of the following values: 0, 1, 2, and 3.

\[
P(X = 0) = \frac{\binom{13}{0} \times \binom{39}{3}}{\binom{52}{3}}
\]

\[
P(X = 1) = \frac{\binom{13}{1} \times \binom{39}{2}}{\binom{52}{3}}
\]

\[
P(X = 2) = \frac{\binom{13}{2} \times \binom{39}{1}}{\binom{52}{3}}
\]

\[
P(X = 3) = \frac{\binom{13}{3} \times \binom{39}{0}}{\binom{52}{3}}
\]

Problem 2

a) Finding \( q \)

First one has to realize that \( q = 0.2 + t \) where \( t \) is the probability that \( X = 0 \).

Now, \( P(X^2 > 0) = 0.8 \Rightarrow 1 - P(X^2 = 0) = 0.8 \Rightarrow P(X^2 = 0) = 0.2 \Leftrightarrow P(X = 0) = 0.2 \). Therefore \( q = 0.2 + 0.2 = 0.4 \).

b) The probability mass function of \( X \) can be written as follows:

\[
P(X) = \begin{cases} 0.2 & x = -1 \\ 0.2 & x = 0 \\ 0.2 & x = 1 \\ 0.1 & x = 2 \\ 0.3 & x = 4 \\ 0 & Otherwise \end{cases}
\]

c) The plot should be a line graph with the values of \( X \) on the abscissa and \( P(X) \) on the ordinate.

d) \( P(X^2 - 1 \geq 3) = P(X^2 \geq 4) = 1 - P(X^2 < 4) = 1 - P(-2 < X < 2) \)
\[ P(X = -1) + P(X = 0) + P(X = 1) = 0.4 \]

e) \( P(2X - 3 \geq 2 | X \geq 0.1) = P(X \geq 2.5 | X \geq 0.1) = \frac{P([X \geq 2.5] \cap (X \geq 0.1))}{P(X \geq 0.1)} \]

\[ \frac{P(X \geq 2.5)}{P(X = 1) + P(X = 2) + P(X = 4)} = \frac{0.3}{0.1 + 0.3 + 0.3} = \frac{3}{7} \]

f) \( E(X) = \sum_{x} xP(X = x) = -1 \times 0.2 + 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.1 + 4 \times 0.3 = 1.4 \)

g) \( E(P(X)) = \sum_{x} P(X = x) \times P(X = x) = 3 \times 0.2^2 + 0.1^2 + 0.3^2 = 0.22 \)

**Problem 3**
a) \( Var(X) = E(X^2) - (E(X))^2 \)

\[ E(X^2) = \int_{20}^{60} x^2 \left( \frac{x - 20}{800} \right) dx = \frac{1}{800} \left( \int_{20}^{60} x^3 dx - 20 \int_{20}^{60} x^2 dx \right) = \frac{6800}{3} \]

\[ E(X) = \int_{20}^{60} \frac{x(x - 20)}{800} dx = \frac{1}{800} \left( \int_{20}^{60} x^2 dx - 20 \int_{20}^{60} x dx \right) = \frac{140}{3} \]

\[ Var(X) = \frac{6800}{3} - \left( \frac{140}{3} \right)^2 = \frac{800}{9} \]

b) \( P(20 \leq X \leq 40) = \int_{20}^{40} \frac{x - 20}{800} dx = \frac{1}{4} \)

c) \( F_y(y) = P(Y < y) = P(2X < y) = P(X < \frac{y}{2}) = F_x(\frac{y}{2}) \)

\[ F_y(y) = \begin{cases} 
0 & y < 40 \\
\int_{20}^{\frac{y}{2}} \frac{x - 20}{800} dx = \frac{y^2 - 80y}{6400} & 40 \leq y \leq 120 \\
1 & y \geq 120 
\end{cases} \]

d)

**Problem 4**

Let \( X \) denote the number of hurricanes per year; \( X \) follows a Poisson(5.2).

\( P(X \leq 3) = \sum_{i=0}^{3} \frac{e^{-5.2} \times (5.2)^{i}}{i!} \)

**Problem 5**
a) Let \( X \) denote the lifetime a radio. It assumed that \( X \) follows an Exp(5)

\( P(X < 5 | X > 2) = 1 - P(X > 5 | X > 2) \). Using the memoryless property of the exponential distribution, we have \( P(X < 5 | X > 2) = 1 - P(X > 3 + 2 | X > 2) = 1 - P(X > 3) = 1 - e^{-15} \).

b) Let’s find the probability that at a radio lasts more than 15 years.

\( P(X > 15) = \int_{15}^{\infty} 5e^{-5x} dx = e^{-75} \).
Let $Y$ be the number of radios that last more than 15 years. $Y$ follows a 
$Binomial(20, e^{-75})$.
We need to compute $P(Y \geq 1)$.

$P(Y \geq 1) = 1 - P(Y = 0)$

$P(Y = 0) = \binom{20}{0} (e^{-75})^0(1 - e^{-75})^{20} \simeq 1$. Therefore, $P(Y \geq 1) \simeq 0$. 