

Basic Principle of Counting:

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if for each outcome of experiment 1 there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Generalized Basic Principle of Counting:

If r that are to be performed are such that the first one may result in n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if ..., then there is a total of $n_1 \times n_2 \times \dots \times n_r$ possible outcomes of the r experiments.

1.3 Permutation

The number of distinguishable permutations of n objects of k types, where n_1 are alike, n_2 are alike, ..., n_k are alike, and $n = n_1 + n_2 + \dots + n_k$ is

$$\frac{n!}{n_1! \times n_2! \times \dots \times n_k!}$$

Example: How many different letter arrangements can be performed using the letter P P E R? Answer: 60.

1.4. Combinations

Definition. An unordered arrangement of r objects from a set A containing n objects ($r < n$) is called an r -element combination of A , or a combination of the elements of A taken r at a time.

There are ${}_n C_r$ combinations of r elements chosen from n objects, where

$${}_n C_r = \frac{n!}{(n-r)!r!}$$

Example. In how many ways can two mathematics and three biology books be selected from eight mathematics and six biology books? Answer 560.

Example. In Maryland's lottery, players pick six different integers between 1 and 49, order of selection being irrelevant. The lottery commission then selects six of these as winning numbers. A player wins the grand prize if all six numbers that he or she has selected match the winning numbers. He or she wins the second prize if exactly five, and the third prize if exactly four of the six numbers chosen match with the winning ones. Find the probability that a certain choice of a bettor wins the grand, the second, and the

third prizes respectively. Answer: first is $1/13,983,816$; second is $258/13,983,816$; and the third prize probability is $13,545/13,983,816$.

Example. Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive? Answer: $\binom{n-m+1}{m}$.

The binomial theorem: For any integer $n > 0$,

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Proof by induction. Second proof by combinatorics.

Example. Evaluate the sum

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}.$$

Answer is $n2^{n-1}$. Note the method of proof for this result; it is a useful trick, that is formalized in the concept of factorial moments.

1.5 Multinomial Coefficients

If $n_1 + n_2 + \dots + n_r = n$, we define $\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \dots n_r!}$. It represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes n_1, n_2, \dots, n_r . It is called a multinomial coefficient.

Example: Ten children are to be divided into an A team and a B team of 5 each. The A team will play in once league and the B team in another. How many different divisions are possible? $\frac{10!}{5!5!}$.