

Indicator Function and its Application in Two-level Factorial Designs

BY KENNY Q. YE

State University of New York at Stony Brook

²A two-level factorial design can be uniquely represented by a polynomial indicator function. Therefore, properties of factorial designs can be studied through their indicator functions. This paper shows that indicator function is an effective tool in studying two-level factorial designs. Indicator function is used to generalize aberration criterion of a regular 2-level fractional factorial design to all 2-level factorial designs. An important identity of generalized aberration is proved. The connection between a uniformity measure and aberration is also extended to all two-level factorial designs.

1. Introduction Two-level factorial designs are most popular designs among experimenters. A 2^{s-k} fractional factorial design is said to be regular if it is generated by k generators. The structure of regular fractional factorial designs are described by group theory and well understood, as discussed in detail by Dey and Mukerjee (1999). Unfortunately, the same theory and mathematical tools can not be applied to non-regular designs and there is not a unified mathematical representation for those designs in general. A recent paper by Fontana *et al.* (2000) introduces

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indicator functions for studying fractional factorial designs (with no replicates). In this paper, indicator functions are extended to study general two-level factorial designs, with or without replicates, regular or non-regular. As to be demonstrated in this paper, it is a very effective tool for studying those designs.

This paper is organized as following. In Section 2, indicator function is defined for all two-level factorial designs. In Section 3, indicator functions are related to aberration of regular 2-level factorial designs. Based on this relation, the definition of aberration is naturally generalized to all two-level factorial designs. In addition, an important property of the generalized criterion is also obtained. In Section 4, indicator function is applied to extend the connection between aberration and a uniformity measure to all two-level factorial designs.

2. Indicator Functions Following the path-breaking paper of Pistone and Wynn (1996) which applies computational algebraic geometry methods to study statistics, Fontana *et al.* (2000) introduce indicator functions as a tool to study fractional factorial designs. Let \mathcal{D} be a 2^s full factorial design with levels being -1 and 1 . The design points of \mathcal{D} are the solutions of the polynomial system $\{x_1^2 - 1 = 0, x_2^2 - 1 = 0, \dots, x_s^2 - 1 = 0\}$. A fractional factorial design \mathcal{F} is a subset of \mathcal{D} . Notice that \mathcal{F} could be any subset of \mathcal{D} without any restriction on its run size. They define *indicator function* of \mathcal{F} as following.

DEFINITION 1. (*Fontana et al. 2000*) Let \mathcal{D} be a 2^s design, the indicator function F of its fraction \mathcal{F} is a function defined on \mathcal{D} such that

$$F(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \mathcal{F} \\ 0 & \text{if } \mathbf{x} \in \mathcal{D} - \mathcal{F} \end{cases}.$$

Using theory of Gröbner basis and algebraic geometry, Fontana *et al.* (2000) show that each indicator function has a unique polynomial representation. A more elementary and constructive but less elegant approach is given below.

The existence of an indicator function in a polynomial form follows immediately from following two lemmas. The proofs of the lemmas are straightforward and are omitted here.

LEMMA 1. *The indicator function of a single point $\mathbf{a} = (a_1, a_2, \dots, a_s) \in \mathcal{D}$ is*

$$(1) \quad F_{\mathbf{a}} = \frac{\prod_{i=1}^s (x_i + a_i)}{\prod_{i=1}^s 2a_i}$$

LEMMA 2. *Let $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ be indicator functions of two disjoint designs \mathcal{A} and \mathcal{B} , respectively. The indicator function of design $\mathcal{A} \cup \mathcal{B}$ is then*

$$(2) \quad F_{\mathcal{A} \cup \mathcal{B}} = F_{\mathcal{A}} + F_{\mathcal{B}}.$$

The uniqueness of the polynomial form can be established as follows. Define contrasts $X_I(\mathbf{x}) = \prod_{i \in I} x_i$ on \mathcal{D} for $I \in \mathcal{P}$, where \mathcal{P} is the collection of all subsets of $\{1, 2, \dots, s\}$. It is well known in experimental design that $\{X_I, I \in \mathcal{P}\}$ forms an orthogonal basis of \mathbb{R}^{2^s} . Therefore, any function f on \mathcal{D} is a unique linear combination of 2^s contrasts. Since each contrast is a monomial on \mathcal{D} , f has a polynomial form

$$(3) \quad f(\mathbf{x}) = \sum_{I \in \mathcal{P}} b_I X_I(\mathbf{x}).$$

Note that since $x_i^2 = 1$ on \mathcal{D} for $i = 1, \dots, s$, the monomials in the polynomial form are “square free”. Therefore, an indicator function has the unique square-free polynomial representation on \mathcal{D} .

The above definition, however, only deals with unreplicated designs. Designs with replicates are often seen in practice and are of theoretical interest as well. For example, Lin and Draper (1992) identify two non-isomorphic 5-column projections of the 12-run Plackett-Burman design. In one of the two projections, a design point repeats twice. In order to study these designs, the definition of indicator functions is generalized as following.

DEFINITION 2. *Let \mathcal{D} be a 2^s design and \mathcal{A} be a design such that $\forall \mathbf{a} \in \mathcal{A}$, $\mathbf{a} \in \mathcal{D}$ but \mathbf{a} might be repeated in \mathcal{A} . The generalized indicator function of \mathcal{A} can be obtained as*

$$F(x_1, \dots, x_s) = \sum_{\mathbf{a} \in \mathcal{A}} F_{\mathbf{a}}(x_1, \dots, x_s),$$

where $F_{\mathbf{a}}$ is the indicator function of point \mathbf{a} .

By this definition, the value of the indicator function of a design \mathcal{A} at point $\mathbf{a} \in \mathcal{A}$ is the number of appearance of \mathbf{a} in \mathcal{A} . By the same argument as for the regular indicator function, a generalized indicator function has a unique polynomial representation as in (3). In the remainder of the paper, I will refer to generalized indicator function as indicator function. The new definition of indicator function allows one to study general two-level factorial designs, with or without replicates. *Note that under the new definition, Lemma 2 holds for any two designs \mathcal{A} and \mathcal{B} and the disjoint condition is no longer needed.*

One of the recent advancements in experimental design is on the hidden projection properties of fractional factorial designs, see Wang and Wu (1995), Cheng (1998). One might be interested in the projections of a design when one or more factors in the original design are no longer considered. The remainder of this section will show that indicator functions of projections can be easily obtained given the

indicator function of the original design. Let \mathcal{B} be a projection of a design \mathcal{A} onto a subset of factors. It contains the same design points of \mathcal{A} but each point only has co-ordinates of the factors that are projected onto. Therefore, projections often have replicated points. The following lemma is obtained directly from the definitions.

LEMMA 3. *Consider design \mathcal{A} . Without loss of generality, let \mathcal{B} be its projection to $\{x_{l+1}, \dots, x_s\}$ and \mathcal{D}^l be the 2^l design on $\{x_1, \dots, x_l\}$. The (generalized) indicator function of \mathcal{B} is then*

$$(4) \quad F_{\mathcal{B}}(x_{l+1}, \dots, x_s) = \sum_{(x_1, \dots, x_l) \in \mathcal{D}^l} F_{\mathcal{A}}(x_1, \dots, x_l, x_{l+1}, \dots, x_s).$$

PROOF. For $\mathbf{a}_2 \notin \mathcal{B}$, by definition, $F_{\mathcal{A}}(\mathbf{a}_1, \mathbf{a}_2) = 0$ for all $\mathbf{a}_1 \in \mathcal{D}^l$. Hence $F_{\mathcal{B}}(\mathbf{a}_2) = 0$. For $\mathbf{a}_2 \in \mathcal{B}$, $F_{\mathcal{A}}(\mathbf{a}_1, \mathbf{a}_2)$ is the number of appearance of $(\mathbf{a}_1, \mathbf{a}_2)$ in \mathcal{A} . Hence, $\sum_{\mathbf{a}_1 \in \mathcal{D}^l} F_{\mathcal{A}}(\mathbf{a}_1, \mathbf{a}_2)$ is the number of appearance of \mathbf{a}_2 in \mathcal{B} . \square

The following Theorem is the same as Proposition 8.2 in Fontana *et al.* (2000) which present it without introducing the generalized indicator functions.

THEOREM 1. *Let \mathcal{A} , \mathcal{B} , \mathcal{D}^l be as defined in Lemma 3. Let $F_{\mathcal{A}} = \sum_{I \in \mathcal{P}} b_I X_I$ be the indicator function of \mathcal{A} . Let \mathcal{P}_1 be the collection of all subsets of $\{l+1, \dots, s\}$. Then*

$$(5) \quad F_{\mathcal{B}}(x_{l+1}, \dots, x_s) = 2^l \sum_{I \in \mathcal{P}_1} b_I X_I.$$

PROOF. Let \mathcal{P}_2 be the collection of all subsets of $\{1, \dots, l\}$. We have

$$(6) \quad F_{\mathcal{A}} = \sum_{I \in \mathcal{P}_1} \left(\sum_{J \in \mathcal{P}_2} b_{I \cup J} X_J \right) X_I.$$

Applying (6) to (4), we have

$$(7) \quad F_{\mathcal{B}}(x_1, \dots, x_l) = \sum_{I \in \mathcal{P}_1} \left(\sum_{J \in \mathcal{P}_2} b_{I \cup J} \left(\sum_{(x_1, \dots, x_l) \in \mathcal{D}^l} X_J \right) \right) X_I$$

From the definition of X_J , it is easy to see that $\sum_{(x_1, \dots, x_l) \in \mathcal{D}^l} X_J = 0$ for all $J \in \mathcal{P}_2 - \{\emptyset\}$ and $\sum_{(x_1, \dots, x_l) \in \mathcal{D}^l} X_{\emptyset} = 2^l$. Equation (5) follows immediately. \square

This section defines the indicator function which is unique for each factorial design. Therefore, it can be used to study the property of a design. In the next section, a connection between the indicator function and aberration of a design is explored.

3. Orthogonality and Aberration Criterion One important design property is orthogonality. A related criterion is aberration criterion. Minimum aberration was first proposed by Fries and Hunter (1980) for regular fractional factorial designs. It is discussed in detail in Wu and Hamada (2000) which also provides a catalogue of minimum aberration designs.

3.1. *Indicator function and orthogonality* Fontana *et al.* (2000) study the orthogonality of designs using indicator functions. Although their concern is only about fractions with no replicates, most of their arguments are still valid for generalized indicator functions and designs with replicates. A key result in their paper links the coefficient of indicator functions to the orthogonality between contrasts.

Consider two contrasts X_I and X_J on the 2^s design \mathcal{D} . X_I and X_J are orthogonal on design \mathcal{D} since $\sum_{\mathbf{x} \in \mathcal{D}} X_I(\mathbf{x})X_J(\mathbf{x}) = 0$ for $I \neq J$. Similarly, the two contrasts are orthogonal on design \mathcal{A} if and only if

$$(8) \quad \sum_{\mathbf{x} \in \mathcal{A}} X_I X_J = 0.$$

Let $F_{\mathcal{A}} = \sum_{I \in \mathcal{P}} b_I X_I$ be the indicator function of \mathcal{A} . Denote the difference between two sets $I \cup J - I \cap J$ by $I \Delta J$. Since $x_i^2 = 1$ on \mathcal{D} , we have $X_I X_J = X_{I \Delta J}$.

The left hand side of the equation (8) can be simplified as follows

$$\begin{aligned} \sum_{\mathbf{x} \in \mathcal{A}} X_I X_J &= \sum_{\mathbf{x} \in \mathcal{D}} F_{\mathcal{A}} X_I X_J = \sum_{\mathbf{x} \in \mathcal{D}} \left(\sum_{K \in \mathcal{P}} b_K X_K \right) X_{I \Delta J} \\ &= \sum_{K \in \mathcal{P}} b_K \sum_{\mathbf{x} \in \mathcal{D}} X_K X_{I \Delta J} = b_{I \Delta J} \sum_{\mathbf{x} \in \mathcal{D}} 1 = 2^s b_{I \Delta J}. \end{aligned}$$

Therefore, X_I and X_J are orthogonal on \mathcal{A} if and only if $b_{I \Delta J} = 0$ in $F_{\mathcal{A}}$. In particular, $b_I = 0$ if and only if X_I is orthogonal to the constant term X_{\emptyset} .

What should be mentioned here are the J -Characteristics used by Deng and Tang (1999) as building blocks in defining their generalized aberration criterion. The J -Characteristics of a design closely relate to coefficients of its indicator function and can be viewed as its orthogonality measure.

DEFINITION 3. (*Deng and Tang, 1999*) *Regard a $n \times s$ design as a set of s columns $\mathcal{A} = \{c_1, c_2, \dots, c_s\}$. For any k -subset $d = \{d_1, d_2, \dots, d_k\}$ of \mathcal{A} , define*

$$J_k(d) = \left| \sum_{i=1}^n d_{i1} \cdots d_{ik} \right|,$$

where d_{ij} is the i th element of column d_j . The $J_k(d)$ values are called the J -Characteristics of \mathcal{A} .

Let $I = \{i_1, i_2, \dots, i_k\}$, and d be the k -subset $\{c_{i_1}, c_{i_2}, \dots, c_{i_k}\}$. From the properties of indicator function of \mathcal{A} ,

$$\begin{aligned} J_k(d) &= \left| \sum_{\mathbf{x} \in \mathcal{A}} X_I \right| = \left| \sum_{\mathbf{x} \in \mathcal{D}} F_{\mathcal{A}} X_I \right| = \left| \sum_{\mathbf{x} \in \mathcal{D}} \sum_{J \in \mathcal{P}} b_J X_J X_I \right| \\ &= \left| \sum_{J \in \mathcal{P}} b_J \sum_{\mathbf{x} \in \mathcal{D}} X_I X_J \right| = \left| b_I \sum_{\mathbf{x} \in \mathcal{D}} X_{\emptyset} \right| = 2^s |b_I|. \end{aligned}$$

Therefore, the coefficient of indicator functions is the same as the signed J -Characteristics up to a constant. Given the signed J -Characteristics of a

design, the design is uniquely determined. It should be noted here that, independently, Tang (2001) also observes the one-to-one relation between a factorial design and its signed J -Characteristics.

3.2. *Aberration of regular fractional factorial designs* Aberration describes the orthogonality of a regular fractional factorial design. Consider a two-level factorial design with s factors. For $I \in \mathcal{P}$, its corresponding contrast X_I is $\prod_{i=1}^s x_i^{w_i}$, where $w_i = 1$ if $i \in I$ and 0 otherwise. Since \mathcal{P} has a one-to-one map to lattice $\{\mathbf{w} = (w_1, w_2, \dots, w_s) \mid w_i = 0, 1\}$, the contrasts can be also denoted as $X^{\mathbf{w}}$. Indicator functions of regular fractions are discussed in detail by Fontana *et al.* (2000). Without loss of generality, consider a regular 2^{s-k} fractional factorial design \mathcal{A} generated by k generating relations, $\{X^{\mathbf{w}^{(1)}} = 1, \dots, X^{\mathbf{w}^{(k)}} = 1\}$ such that $X^{\mathbf{w}^{(i)}} = 1$ for all $x \in \mathcal{A}$. The k generating relations generate a group of defining relations $\{X^{\mathbf{w}} = 1 \mid \mathbf{w} \in W_{\mathcal{A}}\}$, where

$$(9) \quad W_{\mathcal{A}} = \left\{ \mathbf{w} \mid \mathbf{w} = \sum_{i=1}^k u_i \mathbf{w}^{(i)}; u_i = 0, 1; i = 1, \dots, k \right\}.$$

The summation in (9) is as in Galois Field GF(2). It can be easily verified that polynomial form of the indicator function of \mathcal{A} is

$$(10) \quad \begin{aligned} F_{\mathcal{A}} &= \frac{1}{2^k} (X^{\mathbf{w}^{(1)}} + 1)(X^{\mathbf{w}^{(2)}} + 1) \cdots (X^{\mathbf{w}^{(k)}} + 1) \\ &= \frac{1}{2^k} \sum_{\mathbf{w} \in W_{\mathcal{A}}} X^{\mathbf{w}} = \sum_{\mathbf{w} \in W_{\mathcal{A}}} \frac{2^{s-k}}{2^s} X^{\mathbf{w}}. \end{aligned}$$

Note in equation (10), the coefficient $b_{\mathbf{w}}$ of contrast $X^{\mathbf{w}}$ is $\frac{2^{s-k}}{2^s}$ if $\mathbf{w} \in W_{\mathcal{A}}$, 0 otherwise.

A contrast $X^{\mathbf{w}}$ is also called a word with length $\|\mathbf{w}\| = \sum_i^s w_i$. The word length pattern of a regular fractional factorial design \mathcal{A} is defined as $\{\alpha_1(\mathcal{A}), \dots, \alpha_s(\mathcal{A})\}$,

where

$$(11) \quad \alpha_j(\mathcal{A}) = \#\{\mathbf{w} \in W_{\mathcal{A}} \mid \|\mathbf{w}\| = j\}.$$

A connection between word length pattern of a design and its indicator function can be observed from (10) and (11),

$$(12) \quad \alpha_j(\mathcal{A}) = \sum_{\|\mathbf{w}\|=j} \left(\frac{2^s}{n} b_{\mathbf{w}}\right)^2,$$

where n is the number of design points in \mathcal{A} , $b_{\mathbf{w}}$ is the coefficients of $X^{\mathbf{w}}$ in the indicator function. The aberration criterion of regular fractional factorial designs is defined as follows.

DEFINITION 4. *Given two regular fractional factorial designs \mathcal{A}_1 and \mathcal{A}_2 . Design \mathcal{A}_1 is said to have less aberration than \mathcal{A}_2 if there exists r such that*

$$\alpha_1(\mathcal{A}_1) = \alpha_1(\mathcal{A}_2), \dots, \alpha_{r-1}(\mathcal{A}_1) = \alpha_{r-1}(\mathcal{A}_2), \alpha_r(\mathcal{A}_1) < \alpha_r(\mathcal{A}_2).$$

A design has minimum aberration if no other design has less aberration.

The aberration criterion extends the resolution criterion. The latter only compares the length of the shortest defining words while the former compares the numbers of defining words of each length. Therefore, designs with same resolution might be different in aberration criteria. For a detailed discussion, please see Wu and Hamada (2000).

3.3. *Generalized Aberration* A nonregular two-level factorial design \mathcal{A} does not have generating relations and defining relation groups. Therefore, (11) cannot be used to define the word length pattern of a nonregular design. However, one can use (12) to define the word length pattern of any two-level factorial design, regular or nonregular.

DEFINITION 5. Let \mathcal{A} be a two-level factorial design and $F_{\mathcal{A}} = \sum_{I \in \mathcal{P}} b_I X_I$ be its indicator function. Its word length pattern is $\{\alpha_1(\mathcal{A}), \dots, \alpha_s(\mathcal{A})\}$, where $\alpha_j(\mathcal{A}) = \sum_{\|\mathbf{w}\|=j} (\frac{2^s}{n} b_{\mathbf{w}})^2 = \sum_{\#I=j} (\frac{2^s}{n} b_I)^2$.

Based on the generalized definition of word length pattern, the definition of the aberration criterion for regular fractional factorial can be applied to all two-level factorial designs, regular or non-regular, with or without replicates. The next theorem gives an important property of generalized word length pattern which justifies this generalization. The following lemma was given by Fontana *et al.* (2000) for designs without replicates. Nonetheless, their proof applies directly to the general two-level factorial designs.

LEMMA 4. (Fontana *et al.* (2000)) Let \mathcal{A} be a $n \times s$ two-level factorial design. Let $F_{\mathcal{A}} = \sum_{I \in \mathcal{P}} b_I X_I$, then $b_{\emptyset} = n/2^s$.

THEOREM 2. Let \mathcal{A} be a fraction of 2^s design \mathcal{D} with n runs. Its word length pattern $\{\alpha_1(\mathcal{A}), \dots, \alpha_s(\mathcal{A})\}$ is as defined in Definition 5. We have

$$(13) \quad \sum_{j=1}^s \alpha_j(\mathcal{A}) = 2^s (n_2/n^2) - 1,$$

where $n_2 = \sum_{\mathbf{x} \in \mathcal{D}} F_{\mathcal{A}}^2(\mathbf{x})$. For designs with no replicates,

$$(14) \quad \sum_{j=1}^s \alpha_j(\mathcal{A}) = \frac{2^s}{n} - 1.$$

PROOF. Let the indicator function of \mathcal{A} be $F_{\mathcal{A}}(\mathbf{x}) = \sum_{I \in \mathcal{P}} b_I X_I$. Hence,

$$\begin{aligned} \sum_{\mathbf{x} \in \mathcal{D}} F_{\mathcal{A}}^2(\mathbf{x}) &= \sum_{\mathbf{x} \in \mathcal{D}} \left(\sum_{I \in \mathcal{P}} b_I X_I \right)^2 = \sum_{\mathbf{x} \in \mathcal{D}} \sum_{I, J \in \mathcal{P}} b_I b_J X_I(\mathbf{x}) X_J(\mathbf{x}) \\ &= \sum_{I, J \in \mathcal{P}} b_I b_J \sum_{\mathbf{x} \in \mathcal{D}} X_I(\mathbf{x}) X_J(\mathbf{x}) = 2^s \sum_{I \in \mathcal{P}} b_I^2. \end{aligned}$$

From the definition, $\sum_{j=1}^s \alpha_j(\mathcal{A}) = \sum_{I \in \mathcal{P} - \{\emptyset\}} b_I^2 / b_\emptyset^2$. From the lemma, $b_\emptyset = n/2^s$. Hence equation (13) is obtained. For designs with no replicates, $F_{\mathcal{A}}^2(\mathbf{x}) = F_{\mathcal{A}}(\mathbf{x})$ and $n_2 = n$, hence (14) is obtained. \square

For a regular fractional factorial 2^{s-k} design \mathcal{A} , it is well known that fractions $\sum_{j=1}^s \alpha_j(\mathcal{A}) = 2^k - 1$. This identity can be easily obtained through the theory of defining contrast groups. It is, nonetheless, a special case of the above theorem. The sum of word length patterns is constant for all designs with the same replication pattern. For designs with high degree of replication, n_2 is large in (13). Therefore, they tend to have high aberration compared to designs with less replication.

Recently, Deng and Tang (1999) and Tang and Deng (2000) generalize resolution and aberration criterion to nonregular two-level designs based on the J -Characteristics. Coincidentally, the definition of generalized aberration criterion in this paper is equivalent to the G_2 -aberration criterion in Tang and Deng (2000). Without giving a particular reason, Tang and Deng (2000) seem to favor G_2 -aberration criterion over other choice of G_p -aberration which is equivalent to $\alpha_j(\mathcal{A}) = \sum_{\#I=j} \left(\frac{2^s}{n} b_I\right)^p$. Their intuition is well justified by Theorem 2. For other choice of G_p , the sum of word length pattern is no longer constant for designs with same size and same replication pattern.

One application of the above theorem is to search minimum aberration designs using a uniformity criterion which will be discussed later in the paper.

4. Generalized Aberration and Uniformity Uniformity (Space filling property) is another popular design criteria. Fang and Mukerjee (2000) find a connection between aberration and a uniformity measure of regular two-level factorial designs. The uniformity measure that they used is defined in Hickernell (1998).

DEFINITION 6. (*Hickernell*) For design \mathcal{A} as a collection of n design points in $[0, 1]^s$, the centered L_2 -discrepancy measure of uniformity is

$$(15) \quad \begin{aligned} \{CL_2(\mathcal{A})\}^2 &= \left(\frac{13}{12}\right)^s - \frac{2}{n} \sum_{\mathbf{x} \in \mathcal{A}} \prod_{j=1}^s \left(1 + \frac{1}{2}|x_j - 1/2| + \frac{1}{2}|x_j - 1/2|^2\right) \\ &+ \frac{1}{n^2} \sum_{\mathbf{x}, \mathbf{x}' \in \mathcal{A}} \prod_{j=1}^s \left(1 + \frac{1}{2}|x_j - 1/2| + \frac{1}{2}|x'_j - 1/2| - \frac{1}{2}|x_j - x'_j|\right). \end{aligned}$$

Re-scale a two-level factorial design to $[0, 1]^s$ with levels at $1/4$ and $3/4$, a connection between aberration and uniformity is found by Fang and Mukerjee (2000).

THEOREM 3. (*Fang and Mukerjee*) Let \mathcal{A} be a regular design, involving $n = 2^{s-k}$ runs of a 2^s factorial. The centered L_2 -discrepancy measure of uniformity in $[0, 1]^s$

$$(16) \quad \{CL_2(\mathcal{A})\}^2 = \left(\frac{13}{12}\right)^s - 2\left(\frac{35}{32}\right)^s + \left(\frac{9}{8}\right) \left\{1 + \sum_{r=1}^s \frac{\alpha_r(\mathcal{A})}{9^r}\right\}.$$

With definition of generalized word length pattern for non-regular designs, the above theorem extends to all two-level factorial design, regular or nonregular, with or without replicates.

THEOREM 4. Let \mathcal{A} be a $n \times s$ run two-level factorial design. Then L_2 -discrepancy measure of uniformity

$$(17) \quad \{CL_2(\mathcal{A})\}^2 = \left(\frac{13}{12}\right)^s - 2\left(\frac{35}{32}\right)^s + \left(\frac{9}{8}\right) \left\{1 + \sum_{r=1}^s \frac{\alpha_r(\mathcal{A})}{9^r}\right\}.$$

PROOF. The proof follows the same proof in Fang and Mukerjee (2000). Note that “ $\pi(x)'y_d$ ” in their proof equals $2^s b_I$. \square

It can be easily seen from Theorems 2 and 4 that minimum aberration designs should often agree with the most uniform design within the class of all two-level

designs of same run size, regular or non-regular, with replicates or without replicates. Following the geometrical intuition, designs with high degree of replication tend to be less uniform than those with less replicates. This is clearly revealed by Theorems 2 and 4. In general, designs with more replicates are less favored with respect to aberration and uniformity criteria. This provides another justification for the generalized aberration criterion.

Since the uniformity measure in (15) is much easier to compute than word length pattern, it would be a very good searching criterion for minimum aberration design. Note that the first two terms of (15) need not to be computed since they are constant for 2-level designs, and the last term can be simplified to $\sum_{\mathbf{x}, \mathbf{x}' \in \mathcal{A}} \prod_{j=1}^s \left(\frac{5}{4} - \frac{1}{2} |x_j - x'_j| \right)$.

With the uniformity measure as searching criterion, columnwise-pairwise algorithm (Li and Wu, 1997) is used to search minimum aberration designs within balanced 12×5 2-level factorial designs. The search lasted only a few seconds and found the well known subdesign of 12 run Plackett-Burman design with no replicates. Its word length pattern is $(0, 0, 10/9, 5/9, 0)$. The same method was applied for 16-run designs and found designs that matches with those found by Tang and Deng (2000). They construct the designs by searching only projections of the third Hadamard matrix of order 16. While searching within subdesigns of Hadamard matrices is very efficient, it is limited to designs with $4m$ runs. Nonetheless, their search can also be speeded up by using the uniformity criterion. Searching minimum aberration design within all two-level factorial designs of a run size takes longer time but it can be applied to supersaturated designs and designs with $4m + 2$ runs. With columnwise-pairwise algorithm and an easy-to-compute searching criterion, it becomes plausible.

5. Conclusion Traditional factorial design theory focuses on regular fractions. But non-regular designs provide experimenters much more choices, especially more flexible run size. The group theory used to study regular fractions is not suitable for studying general factorial designs. This paper introduces indicator functions as a mathematical representation for general two-level factorial designs. It provides a single mathematical framework for regular and non-regular designs, designs with or without replicates. In this paper, the effectiveness of indicator functions is explored by extending several existing results on regular designs to general factorial designs. First, word length pattern and aberration criterion are generalized to general factorial designs. A justification of such generalization is also provided. Second, a relation between aberration and a uniformity measure is also extended to general factorial designs. The two criteria are very much consistent with each other. Of the same run size, designs with higher degree of replication tend to be less desired with respect to both criteria.

This paper modifies the indicator function introduced by Fontana *et al.* (2000) to represent designs with replicates. One of the referees pointed out another approach to deal with replicates by Cohen *et al.* (2001) which introduces the replication number as a new pseudo-factor. Such arrangement is useful in computing Gröbner basis of the designs with replicates and is complimentary to the indicator function approach proposed here. For details on other application of computational commutative algebra in statistics, see Pistone, *et al.* (2000).

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DEPARTMENT OF APPLIED MATH AND STATISTICS,

SUNY AT STONY BROOK, STONY BROOK, NEW YORK, 11794-3600